The N.P. and G.H.P. formalisms.

An application of *Spinors* and *xCoba.*

Alfonso García-Parrado Gómez-Lobo
Universidade do Minho, Portugal
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In this notebook we derive all the equations of the Newman-Penrose formalism from scratch, by using spinor calculus. We use the equations obtained to introduce the essentials of the GHP formalism (GHP operators, weighted quantities, etc) and then we obtain the set of GHP equations. This notebook can be used as starting point for any computation involving the N.P. and G.H.P. formalisms.

For a quick start, execute sections 1, 2 and 3 and then proceed to section 4 where the essential explanations about how to use this notebook are provided.

■ **1. Spin structure on the spacetime manifold**

We start by loading `Spinors`.

In[1]:= `<< xAct`Spinors``

```
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Package xAct`xPerm`  version 1.0.3, {2009, 9, 9}

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Connecting to external linux executable...

Connection established.

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Package xAct`xTensor`  version 0.9.9, {2009, 9, 14}

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Package xAct`Spinors`  version 0.9.2, {2009, 9, 9}

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```

Definition of a four dimensional manifold M4:

In[2]:= `DefManifold[M4, 4, {a, b, c, d, f, p, q, r, m, l, h, j, n, t, s}]`

** DefManifold: Defining manifold M4.

** DefVBundle: Defining vbundle TangentM4.

Definition of a Lorentzian metric on the manifold M4:

In[3]:= **DefMetric[{3, 1, 0}, g[-a, -b], CD]**

** DefTensor: Defining symmetric metric tensor g[-a, -b].

** DefTensor: Defining antisymmetric tensor epsilong[a, b, c, d].

** DefTensor: Defining tensor Tetrag[-a, -b, -c, -d].

** DefTensor: Defining tensor Tetrag†[-a, -b, -c, -d].

** DefCovD: Defining covariant derivative CD[-a].

** DefTensor: Defining vanishing torsion tensor TorsionCD[a, -b, -c].

** DefTensor: Defining symmetric Christoffel tensor ChristoffelCD[a, -b, -c].

** DefTensor: Defining Riemann tensor RiemannCD[-a, -b, -c, -d].

** DefTensor: Defining symmetric Ricci tensor RicciCD[-a, -b].

** DefCovD: Contractions of Riemann automatically replaced by Ricci.

** DefTensor: Defining Ricci scalar RicciScalarCD[].

** DefCovD: Contractions of Ricci automatically replaced by RicciScalar.

** DefTensor: Defining symmetric Einstein tensor EinsteinCD[-a, -b].

** DefTensor: Defining Weyl tensor WeylCD[-a, -b, -c, -d].

** DefTensor: Defining symmetric TFRicci tensor TFRicciCD[-a, -b].

** DefTensor: Defining Kretschmann scalar KretschmannCD[].

** DefCovD: Computing RiemannToWeylRules for dim 4

** DefCovD: Computing RicciToTFRicci for dim 4

** DefCovD: Computing RicciToEinsteinRules for dim 4

** DefTensor: Defining weight +2 density Detg[]. Determinant.

We define a spin structure comprising the complex vector bundle Spin whose abstract indices are {A, B, C, D, F, H, L, P, Q}. We need to supply as arguments the name of the metric tensor of the background Lorentzian manifold g[-a,-b], the name of the antisymmetric spin metric ϵ[-A,-B], the name of the soldering form σ[a,-A,-A†] and the name of the spin covariant derivative CDe[-a]. The option SpinorPrefix is used to introduce a tag which will be prepended automatically to any spinor related to a spacetime tensor and the option SpinorMark introduces a formatting of the tag SP different to the default one. The long output of DefSpinStructure can be suppressed by adding the option Info->False which deactivates the information of the definition commands.

In[4]:= **DefSpinStructure[g, Spin, {A, B, C, D, F, H, L, P, Q},**
      ϵ, σ, CDe, SpinorPrefix -> SP, SpinorMark -> "S", Info -> False]

    ValidateSymbol::capital : System name C is overloaded as an abstract index.

    ValidateSymbol::capital : System name D is overloaded as an abstract index.

Pretty printing options for the spinor indices and the spin metric.

```
In[5]:= {PrintAs[A†] ^= "A'", PrintAs[B†] ^= "B'", PrintAs[C†] ^= "C'",
          PrintAs[D†] ^= "D'", PrintAs[F†] ^= "F'", PrintAs[H†] ^= "H'", PrintAs[L†] ^= "L'",
          PrintAs[M†] ^= "M'", PrintAs[P†] ^= "P'", PrintAs[Q†] ^= "Q'", PrintAs[ε †] ^= "ε̄"};
```

■ 2. The Newman-Penrose formalism

The Newman-Penrose formalism can be regarded as an application of the spinor calculus. In this section we show how all the equations of this formalism can be derived sytematically and efficiently using `Spinors`` in conjuntion with the package `xCoba``. Full explanations about the usage of `xCoba`` can be found in the package documentation.

We load `xCoba``.

```
In[6]:= << xAct`xCoba`
```

```
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Package xAct`xCoba`  version 0.7.0, {2009, 9, 9}

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```

We turn off the verbose messages issued by `ComponentValue`.

```
In[7]:= $CVVerbose = False;
```

In the following subsections we introduce the spin dyad and its associated null tetrad. We will use `xCoba`` to introduce two frames (basis), called `Dyad` and `NP` and we shall create a parallel set of vectors for the elements in each basis. See the corresponding subsections for full details.

■ Spin dyad

First of all, we define the elements of a spin dyad.

```
In[8]:= DefSpinor[♯, M4] & /@ {o[A], ι[A]};
        PrintAs[o†] ^= "ō";
        PrintAs[ι †] ^= "ῑ";
```

** DefTensor: Defining tensor o[A].

** DefTensor: Defining tensor o†[A†].

** DefTensor: Defining tensor ι[A].

** DefTensor: Defining tensor ι †[A†].

These spinors belong to a spin basis which we define next.

```
In[11]:= DefBasis[Dyad, Spin, {0, 1}, BasisColor → RGBColor[1, 0, 0], Dagger → Complex]
```

```
** DefBasis: Defining basis Dyad.

** DefCovD: Defining parallel derivative PDDyad[-a].

** DefTensor: Defining torsion tensor TorsionPDDyad[a, -b, -c].

** DefTensor: Defining non-symmetric Christoffel tensor ChristoffelPDDyad[a, -b, -c].

** DefTensor: Defining vanishing Riemann tensor RiemannPDDyad[-a, -b, -c, d].

** DefTensor: Defining vanishing Ricci tensor RicciPDDyad[-a, -b].

** DefTensor: Defining nonsymmetric AChristoffel tensor AChristoffelPDDyad[A, -b, -C].

** DefTensor: Defining nonsymmetric AChristoffel tensor AChristoffelPDDyad†[A†, -b, -C†].

** DefTensor: Defining vanishing FRiemann tensor FRiemannPDDyad[-a, -b, -C, D].

** DefTensor: Defining vanishing FRiemann tensor FRiemannPDDyad†[-a, -b, -C†, D†].

** DefBasis: Defining basis Dyad†.

** DefTensor: Defining antisymmetric +1 density etaUpDyad[A, B].

** DefTensor: Defining antisymmetric +1 density etaUpDyad†[A†, B†].

** DefTensor: Defining antisymmetric -1 density etaDownDyad[-A, -B].

** DefTensor: Defining antisymmetric -1 density etaDownDyad†[-A†, -B†].
```

Component indices in the conjugated basis will be also primed :

In[12]:= **Dyad† /: CIndexForm[i_Integer, Dyad†] := StringJoin[ToString[i], "'"];**

By definition, these are the equations relating both sets of objects:

In[13]:= **eqs = {o[A] == Basis[{0, -Dyad}, A], ι[A] == Basis[{1, -Dyad}, A]}**

Out[13]= $\left\{ o^A == e_0{}^A, \ \iota^A == e_1{}^A \right\}$

From them, we read and store the components of the spinors o^A, ι^A .

In[14]:= **Map[ToBasis[Dyad], eqs, {2}]**

Out[14]= $\left\{ o^A == e_0{}^A, \ \iota^A == e_1{}^A \right\}$

We now do the same for the conjugated equations:

In[15]:= **Map[ToBasis[Dyad†], Dagger[eqs], {2}]**

Out[15]= $\left\{ \overline{o}^{A'} == e_{0'}{}^{A'}, \ \overline{\iota}^{A'} == e_{1'}{}^{A'} \right\}$

In[16]:= **ComponentArray@Join[%%, %]**

Out[16]= $\left\{ \left\{ o^0 == 1, o^1 == 0 \right\}, \left\{ \iota^0 == 0, \iota^1 == 1 \right\}, \left\{ \overline{o}^{0'} == 1, \overline{o}^{1'} == 0 \right\}, \left\{ \overline{\iota}^{0'} == 0, \overline{\iota}^{1'} == 1 \right\} \right\}$

In[17]:= **% /. Equal -> ComponentValue;**

Since $\{o^A, \iota^A\}$ is a spin dyad, the spin metric can be written in the form

In[18]:= **eqs = {ϵ[A, B] == o[A] ι[B] - o[B] ι[A]}**

Out[18]= $\left\{ \epsilon^{AB} == -o^B \iota^A + o^A \iota^B \right\}$

The components of the spin metric and its complex conjugate can now be computed and stored.

In[19]:= **Map[ToBasis[Dyad], eqs, {2}]**

Out[19]= $\left\{ \epsilon^{A\,B} == -o^B \iota^A + o^A \iota^B \right\}$

In[20]:= **Map[ToBasis[Dyad†], Dagger[eqs], {2}]**

Out[20]= $\left\{ \overline{\epsilon}^{A'\,B'} == -\overline{o}^{B'} \overline{\iota}^{A'} + \overline{o}^{A'} \overline{\iota}^{B'} \right\}$

In[21]:= **ToValues@ComponentArray@Join[%%, %]**

Out[21]= $\left\{ \left\{ \left\{ \epsilon^{0\,0} == 0, \epsilon^{0\,1} == 1 \right\}, \left\{ \epsilon^{1\,0} == -1, \epsilon^{1\,1} == 0 \right\} \right\}, \left\{ \left\{ \overline{\epsilon}^{0'\,0'} == 0, \overline{\epsilon}^{0'\,1'} == 1 \right\}, \left\{ \overline{\epsilon}^{1'\,0'} == -1, \overline{\epsilon}^{1'\,1'} == 0 \right\} \right\} \right\}$

In[22]:= **% /. Equal -> ComponentValue;**

We deduce from here the components of ϵ_{AB} by means of the relation $\epsilon_{AB}\, \epsilon^{CB} = \delta_A{}^C$. To that end we need to find the inverse of the transposed of the matrix representation of the components of ϵ^{AB}.

In[23]:= **EP = ToValues@TableOfComponents[ϵ[A, B], Dyad]**

Out[23]= {{0, 1}, {-1, 0}}

In[24]:= **Inverse@Transpose@EP**

Out[24]= {{0, 1}, {-1, 0}}

Storage of the components of ϵ^{AB} and its complex conjugate.

In[25]:= **MetricInBasis[ϵ, -Dyad, %];**
MetricInBasis[ϵ †, -Dyad†, %%];

We now compute and store the components of o_A, ι_A and their complex conjugate.

In[27]:= **eqs = {o[-A] == ϵ[-B, -A] o[B], ι[-A] == ϵ[-B, -A] ι[B]}**

Out[27]= $\left\{ o_A == o^B \epsilon_{BA}, \iota_A == \epsilon_{BA} \iota^B \right\}$

In[28]:= **Map[ToBasis[Dyad], eqs, {2}]**

Out[28]= $\left\{ o_A == o^B \epsilon_{B\,A}, \iota_A == \epsilon_{B\,A} \iota^B \right\}$

In[29]:= **Map[ToBasis[Dyad†], Dagger[eqs], {2}]**

Out[29]= $\left\{ \overline{o}_{A'} == \overline{o}^{B'} \overline{\epsilon}_{B'\,A'}, \overline{\iota}_{A'} == \overline{\epsilon}_{B'\,A'} \overline{\iota}^{B'} \right\}$

In[30]:= **Map[TraceBasisDummy, Join[%%, %], {2}]**

Out[30]= $\left\{ o_A == o^0 \epsilon_{0\,A} + o^1 \epsilon_{1\,A}, \iota_A == \epsilon_{0\,A} \iota^0 + \epsilon_{1\,A} \iota^1, \overline{o}_{A'} == \overline{o}^{0'} \overline{\epsilon}_{0'\,A'} + \overline{o}^{1'} \overline{\epsilon}_{1'\,A'}, \overline{\iota}_{A'} == \overline{\epsilon}_{0'\,A'} \overline{\iota}^{0'} + \overline{\epsilon}_{1'\,A'} \overline{\iota}^{1'} \right\}$

In[31]:= **ToValues@ComponentArray@%**

Out[31]= $\left\{ \{o_0 == 0, o_1 == 1\}, \{\iota_0 == -1, \iota_1 == 0\}, \{\overline{o}_{0'} == 0, \overline{o}_{1'} == 1\}, \{\overline{\iota}_{0'} == -1, \overline{\iota}_{1'} == 0\} \right\}$

In[32]:= `% /. Equal -> ComponentValue;`

We change the `Rule` assignment to the `Set` assignment in each `FoldedRule` storing the components computed so far

In[33]:= `RuleToSet /@ {ϵ, ϵ †, o, o†, ι, ι †};`

■ **Null tetrad**

We define now four vectors which will represent the null tetrad basis elements.

In[34]:= `DefTensor[lNP[a], M4, PrintAs → "l"]`

** DefTensor: Defining tensor lNP[a].

In[35]:= `DefTensor[nNP[a], M4, PrintAs → "n"]`

** DefTensor: Defining tensor nNP[a].

In[36]:= `DefTensor[mNP[a], M4, Dagger → Complex, PrintAs → "m"]`

** DefTensor: Defining tensor mNP[a].

** DefTensor: Defining tensor mNP†[a].

In[37]:= `PrintAs@mNP† ^= "m̄";`

These vectors belong to a basis (null tetrad) which we define next.

In[38]:= `DefBasis[NP, TangentM4, {1, 2, 3, 4}, BasisColor -> RGBColor[0, 0, 1]]`

** DefBasis: Defining basis NP.

** DefCovD: Defining parallel derivative PDNP[−a].

** DefTensor: Defining torsion tensor TorsionPDNP[a, −b, −c].

** DefTensor: Defining non−symmetric Christoffel tensor ChristoffelPDNP[a, −b, −c].

** DefTensor: Defining vanishing Riemann tensor RiemannPDNP[−a, −b, −c, d].

** DefTensor: Defining vanishing Ricci tensor RicciPDNP[−a, −b].

** DefTensor: Defining antisymmetric +1 density etaUpNP[a, b, c, d].

** DefTensor: Defining antisymmetric −1 density etaDownNP[−a, −b, −c, −d].

We specify the behaviour of the elements of this basis under complex conjugation. This is achieved with the command `SetDaggerMatrix`.

In[39]:= `SetDaggerMatrix[NP, {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 0, 1}, {0, 0, 1, 0}}]`

We format a number of objects according to standard conventions. Due to the new way of sorting definitions in *Mathematica* 6.0, we need to stop the formatting engine and start it again after defining these new rules.

Stop formatting engine.

In[40]:= `xTensorFormStop[];`

Formatting of the directional derivatives with the standard NP symbols.

In[41]:= **FormatBasis[PDNP[{1, -NP}], "D"];**
FormatBasis[PDNP[{2, -NP}], "Δ"];
FormatBasis[PDNP[{3, -NP}], "δ"];
FormatBasis$\left[$PDNP[{4, -NP}], "$\overline{δ}$"$\right]$;

The Newman-Penrose directional derivatives can also act on spinors. For that case we need to use the covariant derivative CDe rather than PDNP. Therefore we need to set a formatting for the directional derivatives taken with respect to CDe (note that when CDe and PDNP act on scalars or tensors they coincide).

In[45]:= **FormatBasis[CDe[{1, -NP}], "D"];**
FormatBasis[CDe[{2, -NP}], "Δ"];
FormatBasis[CDe[{3, -NP}], "δ"];
FormatBasis$\left[$CDe[{4, -NP}], "$\overline{δ}$"$\right]$;

Restart formatting engine:

In[49]:= **xTensorFormStart[];**

These are the relations between the elements of the null tetrad and the elements of the basis NP.

In[50]:= **{lNP[a] == Basis[a, {1, -NP}], nNP[a] == Basis[a, {2, -NP}],**
mNP[a] == Basis[a, {3, -NP}], mNP†[a] == Basis[a, {4, -NP}]}

Out[50]= $\left\{ l^a == e_1{}^a,\ n^a == e_2{}^a,\ m^a == e_3{}^a,\ \overline{m}^a == e_4{}^a \right\}$

From this relation we compute the components of l^a, n^a, m^a, $\overline{m}^a$ in the basis NP .

In[51]:= **Map[ToBasis[NP], %, {2}]**

Out[51]= $\left\{ l^a == e_1{}^a,\ n^a == e_2{}^a,\ m^a == e_3{}^a,\ \overline{m}^a == e_4{}^a \right\}$

In[52]:= **Flatten@ComponentArray@%**

Out[52]= $\{ l^1 == 1,\ l^2 == 0,\ l^3 == 0,\ l^4 == 0,\ n^1 == 0,\ n^2 == 1,\ n^3 == 0,$
$n^4 == 0,\ m^1 == 0,\ m^2 == 0,\ m^3 == 1,\ m^4 == 0,\ \overline{m}^1 == 0,\ \overline{m}^2 == 0,\ \overline{m}^3 == 0,\ \overline{m}^4 == 1 \}$

In[53]:= **% /. Equal → ComponentValue;**

As is well known, in the null tetrad the metric takes the form

In[54]:= **g[a, b] == lNP[a] nNP[b] + lNP[b] nNP[a] - (mNP[a] mNP†[b] + mNP[b] mNP†[a])**

Out[54]= $g^{ab} == -m^b \overline{m}^a - m^a \overline{m}^b + l^b n^a + l^a n^b$

We compute and store the components of g^{ab} in the basis NP.

In[55]:= **ToBasis[NP] /@ %**

Out[55]= $g^{a\,b} == -m^b \overline{m}^a - m^a \overline{m}^b + l^b n^a + l^a n^b$

In[56]:= **ToValues@ComponentArray@%**

Out[56]= $\left\{ \left\{ g^{1\,1} == 0,\ g^{1\,2} == 1,\ g^{1\,3} == 0,\ g^{1\,4} == 0 \right\},\ \left\{ g^{2\,1} == 1,\ g^{2\,2} == 0,\ g^{2\,3} == 0,\ g^{2\,4} == 0 \right\},\right.$
$\left. \left\{ g^{3\,1} == 0,\ g^{3\,2} == 0,\ g^{3\,3} == 0,\ g^{3\,4} == -1 \right\},\ \left\{ g^{4\,1} == 0,\ g^{4\,2} == 0,\ g^{4\,3} == -1,\ g^{4\,4} == 0 \right\} \right\}$

We store the components of the metric tensor

In[57]:= **% /. Equal -> ComponentValue**

Out[57]= $\left\{ \left\{ g^{1\,1} \to 0,\, g^{1\,2} \to 1,\, g^{1\,3} \to 0,\, g^{1\,4} \to 0 \right\},\, \left\{ g^{2\,1} \to 1,\, g^{2\,2} \to 0,\, g^{2\,3} \to 0,\, g^{2\,4} \to 0 \right\}, \right.$
$\left. \left\{ g^{3\,1} \to 0,\, g^{3\,2} \to 0,\, g^{3\,3} \to 0,\, g^{3\,4} \to -1 \right\},\, \left\{ g^{4\,1} \to 0,\, g^{4\,2} \to 0,\, g^{4\,3} \to -1,\, g^{4\,4} \to 0 \right\} \right\}$

We compute the components of g_{ab} and store their values.

In[58]:= **MatrixForm@ToValues@TableOfComponents[g[a, b], NP]**

Out[58]//MatrixForm=
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

In[59]:= **MatrixForm@Inverse@%**

Out[59]//MatrixForm=
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

In[60]:= **MetricInBasis[g, -NP, %]**

Out[60]= $\{\{ g_{1\,1} \to 0,\, g_{1\,2} \to 1,\, g_{1\,3} \to 0,\, g_{1\,4} \to 0 \},\, \{ g_{2\,1} \to 1,\, g_{2\,2} \to 0,\, g_{2\,3} \to 0,\, g_{2\,4} \to 0 \},$
$\{ g_{3\,1} \to 0,\, g_{3\,2} \to 0,\, g_{3\,3} \to 0,\, g_{3\,4} \to -1 \},\, \{ g_{4\,1} \to 0,\, g_{4\,2} \to 0,\, g_{4\,3} \to -1,\, g_{4\,4} \to 0 \}\}$

Finally we change the Rule assignment to the Set assignment in the FoldedRule storing all the components of the metric tensor.

In[61]:= **RuleToSet@g**

We also need the components for the elements of the null tetrad with the indices downstairs. They are computed from those with the indices upstairs.

In[62]:= **{lNP[-a] == g[-a, -b] lNP[b], nNP[-a] == g[-a, -b] nNP[b],**
 mNP[-a] == g[-a, -b] mNP[b], mNP†[-a] == g[-a, -b] mNP†[b]}

Out[62]= $\left\{ l_a == g_{ab}\, l^b,\, n_a == g_{ab}\, n^b,\, m_a == g_{ab}\, m^b,\, \overline{m}_a == g_{ab}\, \overline{m}^b \right\}$

In[63]:= **Map[ToBasis[NP], %, {2}]**

Out[63]= $\left\{ l_a == g_{a\,b}\, l^b,\, n_a == g_{a\,b}\, n^b,\, m_a == g_{a\,b}\, m^b,\, \overline{m}_a == g_{a\,b}\, \overline{m}^b \right\}$

In[64]:= **Map[TraceBasisDummy, %, {2}]**

Out[64]= $\left\{ l_a == g_{a\,1}\, l^1 + g_{a\,2}\, l^2 + g_{a\,3}\, l^3 + g_{a\,4}\, l^4,\, n_a == g_{a\,1}\, n^1 + g_{a\,2}\, n^2 + g_{a\,3}\, n^3 + g_{a\,4}\, n^4, \right.$
$\left. m_a == g_{a\,1}\, m^1 + g_{a\,2}\, m^2 + g_{a\,3}\, m^3 + g_{a\,4}\, m^4,\, \overline{m}_a == g_{a\,1}\, \overline{m}^1 + g_{a\,2}\, \overline{m}^2 + g_{a\,3}\, \overline{m}^3 + g_{a\,4}\, \overline{m}^4 \right\}$

In[65]:= **ToValues@Flatten@ComponentArray@%**

Out[65]= $\{ l_1 == 0,\, l_2 == 1,\, l_3 == 0,\, l_4 == 0,\, n_1 == 1,\, n_2 == 0,\, n_3 == 0,$
$n_4 == 0,\, m_1 == 0,\, m_2 == 0,\, m_3 == 0,\, m_4 == -1,\, \overline{m}_1 == 0,\, \overline{m}_2 == 0,\, \overline{m}_3 == -1,\, \overline{m}_4 == 0 \}$

In[66]:= **% /. Equal → ComponentValue;**

Automatic replacement of the values of the components of the elements of the null tetrad.

In[67]:= `RuleToSet /@ {lNP, nNP, mNP, mNP†};`

It is convenient to have a rule which relates l^a, n^a, m^a, $\overline{m}^a$ and the elements of the basis NP. We compute such a rule next.

In[68]:= `{lNP[a] == SeparateBasis[NP]@lNP[a], nNP[a] == SeparateBasis[NP]@nNP[a],`
`  mNP[a] == SeparateBasis[NP]@mNP[a], mNP†[a] == SeparateBasis[NP]@mNP†[a]}`

Out[68]= $\left\{ l^a == e_b{}^a\, l^b,\ n^a == e_b{}^a\, n^b,\ m^a == e_b{}^a\, m^b,\ \overline{m}^a == e_b{}^a\, \overline{m}^b \right\}$

In[69]:= `Map[TraceBasisDummy, %, {2}]`

Out[69]= $\left\{ l^a == e_1{}^a,\ n^a == e_2{}^a,\ m^a == e_3{}^a,\ \overline{m}^a == e_4{}^a \right\}$

In[70]:= `Solve[%, Cases[%, Basis[{__}, a_], Infinity]] // Flatten`

Out[70]= $\left\{ e_1{}^a \to l^a,\ e_2{}^a \to n^a,\ e_3{}^a \to m^a,\ e_4{}^a \to \overline{m}^a \right\}$

In[71]:= `% /. HoldPattern@Rule[X_, Y_] :> MakeRule[{X, Y}] // Flatten // Union`

Out[71]= $\left\{ \text{HoldPattern}\!\left[e_1{}^{\overline{a}}\right] :\to \text{Module}[\{\}, l^a],\ \text{HoldPattern}\!\left[e_2{}^{\overline{a}}\right] :\to \text{Module}[\{\}, n^a], \right.$
$\left. \text{HoldPattern}\!\left[e_3{}^{\overline{a}}\right] :\to \text{Module}[\{\}, m^a],\ \text{HoldPattern}\!\left[e_4{}^{\overline{a}}\right] :\to \text{Module}[\{\}, \overline{m}^a] \right\}$

In[72]:= `BasisToVectors = %;`

We need the same rule with the indices downstairs.

In[73]:= `{lNP[-a] == SeparateBasis[NP]@lNP[-a], nNP[-a] == SeparateBasis[NP]@nNP[-a],`
`  mNP[-a] == SeparateBasis[NP]@mNP[-a], mNP†[-a] == SeparateBasis[NP]@mNP†[-a]}`

Out[73]= $\left\{ l_a == e_a{}^b\, l_b,\ n_a == e_a{}^b\, n_b,\ m_a == e_a{}^b\, m_b,\ \overline{m}_a == e_a{}^b\, \overline{m}_b \right\}$

In[74]:= `Map[TraceBasisDummy, %, {2}]`

Out[74]= $\left\{ l_a == e_a{}^2,\ n_a == e_a{}^1,\ m_a == -e_a{}^4,\ \overline{m}_a == -e_a{}^3 \right\}$

In[75]:= `Solve[%, Cases[%, Basis[__], Infinity]] // Flatten`

Out[75]= $\left\{ e_a{}^2 \to l_a,\ e_a{}^1 \to n_a,\ e_a{}^4 \to -m_a,\ e_a{}^3 \to -\overline{m}_a \right\}$

In[76]:= `% /. HoldPattern@Rule[X_, Y_] :> MakeRule[{X, Y}] // Flatten // Union`

Out[76]= $\left\{ \text{HoldPattern}\!\left[e^{\overline{a}\,1}\right] :\to \text{Module}[\{\}, n^a],\ \text{HoldPattern}\!\left[e^{\overline{a}\,2}\right] :\to \text{Module}[\{\}, l^a], \right.$
$\left. \text{HoldPattern}\!\left[e^{\overline{a}\,3}\right] :\to \text{Module}[\{\}, -\overline{m}^a],\ \text{HoldPattern}\!\left[e^{\overline{a}\,4}\right] :\to \text{Module}[\{\}, -m^a] \right\}$

We combine these relations with the relations found above for the indices upstairs.

In[77]:= `AppendTo[BasisToVectors, %]`

Out[77]= $\left\{ \text{HoldPattern}\!\left[e_1{}^{\overline{a}}\right] :\to \text{Module}[\{\}, l^a],\ \text{HoldPattern}\!\left[e_2{}^{\overline{a}}\right] :\to \text{Module}[\{\}, n^a], \right.$
$\text{HoldPattern}\!\left[e_3{}^{\overline{a}}\right] :\to \text{Module}[\{\}, m^a],\ \text{HoldPattern}\!\left[e_4{}^{\overline{a}}\right] :\to \text{Module}[\{\}, \overline{m}^a],$
$\left\{ \text{HoldPattern}\!\left[e^{\overline{a}\,1}\right] :\to \text{Module}[\{\}, n^a],\ \text{HoldPattern}\!\left[e^{\overline{a}\,2}\right] :\to \text{Module}[\{\}, l^a], \right.$
$\left. \left. \text{HoldPattern}\!\left[e^{\overline{a}\,3}\right] :\to \text{Module}[\{\}, -\overline{m}^a],\ \text{HoldPattern}\!\left[e^{\overline{a}\,4}\right] :\to \text{Module}[\{\}, -m^a] \right\} \right\}$

Relation between the spin dyad and the null tetrad

The relation between the elements of the spinor dyad and the elements of the null tetrad is as follows

In[78]:= `SpinDyadToNullTetrad =`
`  {σ[a, -A, -A†] o[A] Dagger@o[A] == lNP[a], σ[a, -A, -A†] ι[A] Dagger@ι[A] == nNP[a],`
`   σ[a, -A, -A†] o[A] Dagger@ι[A] == mNP[a], σ[a, -A, -A†] Dagger[o[A] Dagger@ι[A]] == mNP†[a]}`

Out[78]= $\left\{ o^A \overline{o}^{A'} \sigma^a{}_{AA'} == l^a, \ \iota^A \overline{\iota}^{A'} \sigma^a{}_{AA'} == n^a, \ o^A \overline{\iota}^{A'} \sigma^a{}_{AA'} == m^a, \ \overline{o}^{A'} \iota^A \sigma^a{}_{AA'} == \overline{m}^a \right\}$

Using this relation we find and store the components of the soldering form $\sigma^a{}_{AA'}$.

In[79]:= `Map[ToBasis[Dyad], %, {2}]`

Out[79]= $\left\{ o^A \overline{o}^{A'} \sigma^a{}_{AA'} == l^a, \ \iota^A \overline{\iota}^{A'} \sigma^a{}_{AA'} == n^a, \ o^A \overline{\iota}^{A'} \sigma^a{}_{AA'} == m^a, \ \overline{o}^{A'} \iota^A \sigma^a{}_{AA'} == \overline{m}^a \right\}$

In[80]:= `Map[ToBasis[Dyad†], %, {2}]`

Out[80]= $\left\{ o^A \overline{o}^{A'} \sigma^a{}_{AA'} == l^a, \ \iota^A \overline{\iota}^{A'} \sigma^a{}_{AA'} == n^a, \ o^A \overline{\iota}^{A'} \sigma^a{}_{AA'} == m^a, \ \overline{o}^{A'} \iota^A \sigma^a{}_{AA'} == \overline{m}^a \right\}$

In[81]:= `Map[ToBasis[NP], %, {2}]`

Out[81]= $\left\{ o^A \overline{o}^{A'} \sigma^a{}_{AA'} == l^a, \ \iota^A \overline{\iota}^{A'} \sigma^a{}_{AA'} == n^a, \ o^A \overline{\iota}^{A'} \sigma^a{}_{AA'} == m^a, \ \overline{o}^{A'} \iota^A \sigma^a{}_{AA'} == \overline{m}^a \right\}$

In[82]:= `Map[TraceBasisDummy, %, {2}]`

Out[82]= $\left\{ \sigma^a{}_{00'} == l^a, \ \sigma^a{}_{11'} == n^a, \ \sigma^a{}_{01'} == m^a, \ \sigma^a{}_{10'} == \overline{m}^a \right\}$

In[83]:= `ComponentArray@%`

Out[83]= $\left\{ \left\{ \sigma^1{}_{00'} == 1, \ \sigma^2{}_{00'} == 0, \ \sigma^3{}_{00'} == 0, \ \sigma^4{}_{00'} == 0 \right\}, \ \left\{ \sigma^1{}_{11'} == 0, \ \sigma^2{}_{11'} == 1, \ \sigma^3{}_{11'} == 0, \ \sigma^4{}_{11'} == 0 \right\}, \right.$
$\left. \left\{ \sigma^1{}_{01'} == 0, \ \sigma^2{}_{01'} == 0, \ \sigma^3{}_{01'} == 1, \ \sigma^4{}_{01'} == 0 \right\}, \ \left\{ \sigma^1{}_{10'} == 0, \ \sigma^2{}_{10'} == 0, \ \sigma^3{}_{10'} == 0, \ \sigma^4{}_{10'} == 1 \right\} \right\}$

In[84]:= `% /. Equal → ComponentValue`

Out[84]= $\left\{ \left\{ \sigma^1{}_{00'} \to 1, \ \sigma^2{}_{00'} \to 0, \ \sigma^3{}_{00'} \to 0, \ \sigma^4{}_{00'} \to 0 \right\}, \ \left\{ \sigma^1{}_{11'} \to 0, \ \sigma^2{}_{11'} \to 1, \ \sigma^3{}_{11'} \to 0, \ \sigma^4{}_{11'} \to 0 \right\}, \right.$
$\left. \left\{ \sigma^1{}_{01'} \to 0, \ \sigma^2{}_{01'} \to 0, \ \sigma^3{}_{01'} \to 1, \ \sigma^4{}_{01'} \to 0 \right\}, \ \left\{ \sigma^1{}_{10'} \to 0, \ \sigma^2{}_{10'} \to 0, \ \sigma^3{}_{10'} \to 0, \ \sigma^4{}_{10'} \to 1 \right\} \right\}$

We also need the components of the soldering form with the spacetime index downstairs and the spin indices upstairs.

In[85]:= `NullTetradToSpinDyad = Map[PutSolderingForm, SpinDyadToNullTetrad, {2}] /. B -> A /. B† -> A†`

Out[85]= $\left\{ o^A \overline{o}^{A'} == l^a \sigma_a{}^{AA'}, \ \iota^A \overline{\iota}^{A'} == n^a \sigma_a{}^{AA'}, \ o^A \overline{\iota}^{A'} == m^a \sigma_a{}^{AA'}, \ \overline{o}^{A'} \iota^A == \overline{m}^a \sigma_a{}^{AA'} \right\}$

In[86]:= `Map[ToBasis[Dyad], %, {2}]`

Out[86]= $\left\{ o^A \overline{o}^{A'} == l^a \sigma_a{}^{AA'}, \ \iota^A \overline{\iota}^{A'} == n^a \sigma_a{}^{AA'}, \ o^A \overline{\iota}^{A'} == m^a \sigma_a{}^{AA'}, \ \overline{o}^{A'} \iota^A == \overline{m}^a \sigma_a{}^{AA'} \right\}$

In[87]:= `Map[ToBasis[Dyad†], %, {2}]`

Out[87]= $\left\{ o^A \overline{o}^{A'} == l^a \sigma_a{}^{AA'}, \ \iota^A \overline{\iota}^{A'} == n^a \sigma_a{}^{AA'}, \ o^A \overline{\iota}^{A'} == m^a \sigma_a{}^{AA'}, \ \overline{o}^{A'} \iota^A == \overline{m}^a \sigma_a{}^{AA'} \right\}$

In[88]:= `Map[ToBasis[NP], %, {2}]`

Out[88]= $\left\{ o^A \overline{o}^{A'} == l^a \sigma_a{}^{AA'}, \ \iota^A \overline{\iota}^{A'} == n^a \sigma_a{}^{AA'}, \ o^A \overline{\iota}^{A'} == m^a \sigma_a{}^{AA'}, \ \overline{o}^{A'} \iota^A == \overline{m}^a \sigma_a{}^{AA'} \right\}$

In[89]:= **Map[TraceBasisDummy, %, {2}]**

Out[89]= $\left\{ o^A \, \overline{o}^{A'} == \sigma_1{}^{A\,A'} ,\; \iota^A \, \overline{\iota}^{A'} == \sigma_2{}^{A\,A'} ,\; o^A \, \overline{\iota}^{A'} == \sigma_3{}^{A\,A'} ,\; \overline{o}^{A'} \, \iota^A == \sigma_4{}^{A\,A'} \right\}$

In[90]:= **Flatten@ComponentArray@%**

Out[90]= $\left\{ 1 == \sigma_1{}^{0\,0'} ,\; 0 == \sigma_1{}^{0\,1'} ,\; 0 == \sigma_1{}^{1\,0'} ,\; 0 == \sigma_1{}^{1\,1'} ,\; 0 == \sigma_2{}^{0\,0'} ,\; 0 == \sigma_2{}^{0\,1'} ,\; 0 == \sigma_2{}^{1\,0'} ,\; 1 == \sigma_2{}^{1\,1'} ,\right.$
$\left. 0 == \sigma_3{}^{0\,0'} ,\; 1 == \sigma_3{}^{0\,1'} ,\; 0 == \sigma_3{}^{1\,0'} ,\; 0 == \sigma_3{}^{1\,1'} ,\; 0 == \sigma_4{}^{0\,0'} ,\; 0 == \sigma_4{}^{0\,1'} ,\; 1 == \sigma_4{}^{1\,0'} ,\; 0 == \sigma_4{}^{1\,1'} \right\}$

In[91]:= **Solve[%, Cases[%, _σ, Infinity]]**

Out[91]= $\left\{ \left\{ \sigma_1{}^{0\,0'} \to 1,\; \sigma_1{}^{0\,1'} \to 0,\; \sigma_1{}^{1\,0'} \to 0,\; \sigma_1{}^{1\,1'} \to 0,\; \sigma_2{}^{0\,0'} \to 0,\; \sigma_2{}^{0\,1'} \to 0,\; \sigma_2{}^{1\,0'} \to 0,\; \sigma_2{}^{1\,1'} \to 1, \right. \right.$
$\left. \left. \sigma_3{}^{0\,0'} \to 0,\; \sigma_3{}^{0\,1'} \to 1,\; \sigma_3{}^{1\,0'} \to 0,\; \sigma_3{}^{1\,1'} \to 0,\; \sigma_4{}^{0\,0'} \to 0,\; \sigma_4{}^{0\,1'} \to 0,\; \sigma_4{}^{1\,0'} \to 1,\; \sigma_4{}^{1\,1'} \to 0 \right\} \right\}$

We store these values.

In[92]:= **% /. Rule → ComponentValue**

Out[92]= $\left\{ \left\{ \sigma_1{}^{0\,0'} \to 1,\; \sigma_1{}^{0\,1'} \to 0,\; \sigma_1{}^{1\,0'} \to 0,\; \sigma_1{}^{1\,1'} \to 0,\; \sigma_2{}^{0\,0'} \to 0,\; \sigma_2{}^{0\,1'} \to 0,\; \sigma_2{}^{1\,0'} \to 0,\; \sigma_2{}^{1\,1'} \to 1, \right. \right.$
$\left. \left. \sigma_3{}^{0\,0'} \to 0,\; \sigma_3{}^{0\,1'} \to 1,\; \sigma_3{}^{1\,0'} \to 0,\; \sigma_3{}^{1\,1'} \to 0,\; \sigma_4{}^{0\,0'} \to 0,\; \sigma_4{}^{0\,1'} \to 0,\; \sigma_4{}^{1\,0'} \to 1,\; \sigma_4{}^{1\,1'} \to 0 \right\} \right\}$

Change of the `Rule` assignment to the `Set` assignment in the folded rule.

In[93]:= **RuleToSet@σ**

■ **Newman-Penrose spin coefficients**

We define the 12 Newmann-Penrose spin coefficients, and their complex conjugates:

In[94]:= **DefSpinor[κ Dyad[], M4];**
PrintAs[κ Dyad] ^= "κ";
PrintAs[κ Dyad†] ^= "$\overline{κ}$";

** DefTensor: Defining tensor κ Dyad[].

** DefTensor: Defining tensor κ Dyad†[].

In[97]:= **DefSpinor[ϵ Dyad[], M4];**
PrintAs[ϵ Dyad] ^= "ϵ";
PrintAs[ϵ Dyad†] ^= "$\overline{ϵ}$";

** DefTensor: Defining tensor ϵ Dyad[].

** DefTensor: Defining tensor ϵ Dyad†[].

In[100]:= **DefSpinor[π Dyad[], M4];**
PrintAs[π Dyad] ^= "π";
PrintAs[π Dyad†] ^= "$\overline{π}$";

** DefTensor: Defining tensor π Dyad[].

** DefTensor: Defining tensor π Dyad†[].

In[103]:= **DefSpinor[τ Dyad[], M4];**
PrintAs[τ Dyad] ^= "τ";
PrintAs[τ Dyad†] ^= "$\overline{τ}$";

** DefTensor: Defining tensor τ Dyad[].

** DefTensor: Defining tensor τ Dyad†[].

```
In[106]:= DefSpinor[γ Dyad[]], M4];
       PrintAs[γ Dyad] ^= "γ";
       PrintAs[γ Dyad†]] ^= "γ̄";
```

** DefTensor: Defining tensor γ Dyad[].

** DefTensor: Defining tensor γ Dyad†[].

```
In[109]:= DefSpinor[ν Dyad[]], M4];
       PrintAs[ν Dyad] ^= "ν";
       PrintAs[ν Dyad†]] ^= "ν̄";
```

** DefTensor: Defining tensor ν Dyad[].

** DefTensor: Defining tensor ν Dyad†[].

```
In[112]:= DefSpinor[σ Dyad[]], M4];
       PrintAs[σ Dyad] ^= "σ";
       PrintAs[σ Dyad†]] ^= "σ̄";
```

** DefTensor: Defining tensor σ Dyad[].

** DefTensor: Defining tensor σ Dyad†[].

```
In[115]:= DefSpinor[β Dyad[]], M4];
       PrintAs[β Dyad] ^= "β";
       PrintAs[β Dyad†]] ^= "β̄";
```

** DefTensor: Defining tensor β Dyad[].

** DefTensor: Defining tensor β Dyad†[].

```
In[118]:= DefSpinor[μ Dyad[]], M4];
       PrintAs[μ Dyad] ^= "μ";
       PrintAs[μ Dyad†]] ^= "μ̄";
```

** DefTensor: Defining tensor μ Dyad[].

** DefTensor: Defining tensor μ Dyad†[].

```
In[121]:= DefSpinor[ρ Dyad[]], M4];
       PrintAs[ρ Dyad] ^= "ρ";
       PrintAs[ρ Dyad†]] ^= "ρ̄";
```

** DefTensor: Defining tensor ρ Dyad[].

** DefTensor: Defining tensor ρ Dyad†[].

```
In[124]:= DefSpinor[α Dyad[]], M4];
       PrintAs[α Dyad] ^= "α";
       PrintAs[α Dyad†]] ^= "ᾱ";
```

** DefTensor: Defining tensor α Dyad[].

** DefTensor: Defining tensor α Dyad†[].

```
In[127]:= DefSpinor[λ Dyad[]], M4];
       PrintAs[λ Dyad] ^= "λ";
       PrintAs[λ Dyad†]] ^= "λ̄";
```

** DefTensor: Defining tensor λ Dyad[].

** DefTensor: Defining tensor λ Dyad†[].

This is full the list of Newman-Penrose spin coefficients.

In[130]:= `NPSpinCoefficients = {α Dyad[], β Dyad[], γ Dyad[], ε Dyad[],`
`κ Dyad[], λ Dyad[], μ Dyad[], ν Dyad[], π Dyad[], ρ Dyad[], σ Dyad[], τ Dyad[]}`

Out[130]= $\{\alpha\ ,\ \beta\ ,\ \gamma\ ,\ \epsilon\ ,\ \kappa\ ,\ \lambda\ ,\ \mu\ ,\ \nu\ ,\ \pi\ ,\ \rho\ ,\ \sigma\ ,\ \tau\ \}$

In[131]:= `% // Length`

Out[131]= 12

The spin coefficients are defined through the derivations of the elements of the spin tetrad by means of the following relations.

In[132]:= `DyadDerivatives = {`
`    CDe[{1, -NP}][o[-A]] == o[-A] ε Dyad[] - ι[-A] κ Dyad[],`
`    CDe[{1, -NP}][ι[-A]] == -ι[-A] ε Dyad[] + o[-A] π Dyad[],`
`    CDe[{2, -NP}][o[-A]] == o[-A] γ Dyad[] - ι[-A] τ Dyad[],`
`    CDe[{2, -NP}][ι[-A]] == -ι[-A] γ Dyad[] + o[-A] ν Dyad[],`
`    CDe[{3, -NP}][o[-A]] == o[-A] β Dyad[] - ι[-A] σ Dyad[],`
`    CDe[{3, -NP}][ι[-A]] == -ι[-A] β Dyad[] + o[-A] μ Dyad[],`
`    CDe[{4, -NP}][o[-A]] == o[-A] α Dyad[] - ι[-A] ρ Dyad[],`
`    CDe[{4, -NP}][ι[-A]] == -ι[-A] α Dyad[] + o[-A] λ Dyad[]`
`    }`

Out[132]= $\Big\{ D\ o_A == o_A\ \epsilon\ -\ \iota_A\ \kappa\ ,\ D\ \iota_A == -\epsilon\ \ \iota_A + o_A\ \pi\ ,\ \triangle\ o_A == o_A\ \gamma\ -\ \iota_A\ \tau\ ,\ \triangle\ \iota_A == -\gamma\ \ \iota_A + o_A\ \nu\ ,$
$\delta\ o_A == o_A\ \beta\ -\ \iota_A\ \sigma\ ,\ \delta\ \iota_A == -\beta\ \ \iota_A + o_A\ \mu\ ,\ \overline{\delta}\ o_A == o_A\ \alpha\ -\ \iota_A\ \rho\ ,\ \overline{\delta}\ \iota_A == -\alpha\ \ \iota_A + o_A\ \lambda\ \Big\}$

From this expression we compute and store the components of the spin covariant derivative "intrinsic" connection (these are the components of the "inner" Christoffel symbols of the covariant derivative `CDe`).

In[133]:= `Map[ToBasis[Dyad], %, {2}]`

Out[133]= $\Big\{ e_A{}^B\ (D\ o_B) == o_A\ \epsilon\ -\ \iota_A\ \kappa\ ,\ e_A{}^B\ (D\ \iota_B) == -\epsilon\ \ \iota_A + o_A\ \pi\ ,$
$e_A{}^B\ (\triangle\ o_B) == o_A\ \gamma\ -\ \iota_A\ \tau\ ,\ e_A{}^B\ (\triangle\ \iota_B) == -\gamma\ \ \iota_A + o_A\ \nu\ ,\ e_A{}^B\ (\delta\ o_B) == o_A\ \beta\ -\ \iota_A\ \sigma\ ,$
$e_A{}^B\ (\delta\ \iota_B) == -\beta\ \ \iota_A + o_A\ \mu\ ,\ e_A{}^B\ (\overline{\delta}\ o_B) == o_A\ \alpha\ -\ \iota_A\ \rho\ ,\ e_A{}^B\ (\overline{\delta}\ \iota_B) == -\alpha\ \ \iota_A + o_A\ \lambda\ \Big\}$

In[134]:= `Map[ToBasis[Dyad], %, {2}]`

** DefTensor: Defining tensor AChristoffelCDePDDyad[A, -b, -C].

** DefTensor: Defining tensor AChristoffelCDePDDyad†[A†, -b, -C†].

Out[134]= $\Big\{ -A[\nabla,\mathcal{D}]^B{}_{1A}\ o_B + D\ o_A == o_A\ \epsilon\ -\ \iota_A\ \kappa\ ,\ -A[\nabla,\mathcal{D}]^B{}_{1A}\ \iota_B + D\ \iota_A == -\epsilon\ \ \iota_A + o_A\ \pi\ ,$
$-A[\nabla,\mathcal{D}]^B{}_{2A}\ o_B + \triangle\ o_A == o_A\ \gamma\ -\ \iota_A\ \tau\ ,\ -A[\nabla,\mathcal{D}]^B{}_{2A}\ \iota_B + \triangle\ \iota_A == -\gamma\ \ \iota_A + o_A\ \nu\ ,$
$-A[\nabla,\mathcal{D}]^B{}_{3A}\ o_B + \delta\ o_A == o_A\ \beta\ -\ \iota_A\ \sigma\ ,\ -A[\nabla,\mathcal{D}]^B{}_{3A}\ \iota_B + \delta\ \iota_A == -\beta\ \ \iota_A + o_A\ \mu\ ,$
$-A[\nabla,\mathcal{D}]^B{}_{4A}\ o_B + \overline{\delta}\ o_A == o_A\ \alpha\ -\ \iota_A\ \rho\ ,\ -A[\nabla,\mathcal{D}]^B{}_{4A}\ \iota_B + \overline{\delta}\ \iota_A == -\alpha\ \ \iota_A + o_A\ \lambda\ \Big\}$

In[135]:= `Map[TraceBasisDummy, %, {2}]`

Out[135]= $\Big\{ -A[\nabla,\mathcal{D}]^1{}_{1A} + D\ o_A == o_A\ \epsilon\ -\ \iota_A\ \kappa\ ,\ A[\nabla,\mathcal{D}]^0{}_{1A} + D\ \iota_A == -\epsilon\ \ \iota_A + o_A\ \pi\ ,$
$-A[\nabla,\mathcal{D}]^1{}_{2A} + \triangle\ o_A == o_A\ \gamma\ -\ \iota_A\ \tau\ ,\ A[\nabla,\mathcal{D}]^0{}_{2A} + \triangle\ \iota_A == -\gamma\ \ \iota_A + o_A\ \nu\ ,\ -A[\nabla,\mathcal{D}]^1{}_{3A} + \delta\ o_A == o_A\ \beta\ -\ \iota_A\ \sigma\ ,$
$A[\nabla,\mathcal{D}]^0{}_{3A} + \delta\ \iota_A == -\beta\ \ \iota_A + o_A\ \mu\ ,\ -A[\nabla,\mathcal{D}]^1{}_{4A} + \overline{\delta}\ o_A == o_A\ \alpha\ -\ \iota_A\ \rho\ ,\ A[\nabla,\mathcal{D}]^0{}_{4A} + \overline{\delta}\ \iota_A == -\alpha\ \ \iota_A + o_A\ \lambda\ \Big\}$

In[136]:= `Flatten@ComponentArray@%`

Out[136]= $\Big\{ -A[\nabla,\mathcal{D}]^1{}_{10} == \kappa\ ,\ -A[\nabla,\mathcal{D}]^1{}_{11} == \epsilon\ ,\ A[\nabla,\mathcal{D}]^0{}_{10} == \epsilon\ ,\ A[\nabla,\mathcal{D}]^0{}_{11} == \pi\ ,\ -A[\nabla,\mathcal{D}]^1{}_{20} == \tau\ ,$
$-A[\nabla,\mathcal{D}]^1{}_{21} == \gamma\ ,\ A[\nabla,\mathcal{D}]^0{}_{20} == \gamma\ ,\ A[\nabla,\mathcal{D}]^0{}_{21} == \nu\ ,\ -A[\nabla,\mathcal{D}]^1{}_{30} == \sigma\ ,\ -A[\nabla,\mathcal{D}]^1{}_{31} == \beta\ ,$
$A[\nabla,\mathcal{D}]^0{}_{30} == \beta\ ,\ A[\nabla,\mathcal{D}]^0{}_{31} == \mu\ ,\ -A[\nabla,\mathcal{D}]^1{}_{40} == \rho\ ,\ -A[\nabla,\mathcal{D}]^1{}_{41} == \alpha\ ,\ A[\nabla,\mathcal{D}]^0{}_{40} == \alpha\ ,\ A[\nabla,\mathcal{D}]^0{}_{41} == \lambda\ \Big\}$

In[137]:= `Solve[%, Cases[%, _AChristoffelCDePDDyad, Infinity]] // Flatten`

Out[137]= $\{A[\nabla,\mathcal{D}]^1{}_{1\,0} \to -\kappa$, $A[\nabla,\mathcal{D}]^1{}_{1\,1} \to -\epsilon$, $A[\nabla,\mathcal{D}]^0{}_{1\,0} \to \epsilon$, $A[\nabla,\mathcal{D}]^0{}_{1\,1} \to \pi$, $A[\nabla,\mathcal{D}]^1{}_{2\,0} \to -\tau$,

$A[\nabla,\mathcal{D}]^1{}_{2\,1} \to -\gamma$, $A[\nabla,\mathcal{D}]^0{}_{2\,0} \to \gamma$, $A[\nabla,\mathcal{D}]^0{}_{2\,1} \to \nu$, $A[\nabla,\mathcal{D}]^1{}_{3\,0} \to -\sigma$, $A[\nabla,\mathcal{D}]^1{}_{3\,1} \to -\beta$,

$A[\nabla,\mathcal{D}]^0{}_{3\,0} \to \beta$, $A[\nabla,\mathcal{D}]^0{}_{3\,1} \to \mu$, $A[\nabla,\mathcal{D}]^1{}_{4\,0} \to -\rho$, $A[\nabla,\mathcal{D}]^1{}_{4\,1} \to -\alpha$, $A[\nabla,\mathcal{D}]^0{}_{4\,0} \to \alpha$, $A[\nabla,\mathcal{D}]^0{}_{4\,1} \to \lambda \}$

In[138]:= `% /. Rule -> ComponentValue`

Out[138]= $\{A[\nabla,\mathcal{D}]^1{}_{1\,0} \to -\kappa$, $A[\nabla,\mathcal{D}]^1{}_{1\,1} \to -\epsilon$, $A[\nabla,\mathcal{D}]^0{}_{1\,0} \to \epsilon$, $A[\nabla,\mathcal{D}]^0{}_{1\,1} \to \pi$, $A[\nabla,\mathcal{D}]^1{}_{2\,0} \to -\tau$,

$A[\nabla,\mathcal{D}]^1{}_{2\,1} \to -\gamma$, $A[\nabla,\mathcal{D}]^0{}_{2\,0} \to \gamma$, $A[\nabla,\mathcal{D}]^0{}_{2\,1} \to \nu$, $A[\nabla,\mathcal{D}]^1{}_{3\,0} \to -\sigma$, $A[\nabla,\mathcal{D}]^1{}_{3\,1} \to -\beta$,

$A[\nabla,\mathcal{D}]^0{}_{3\,0} \to \beta$, $A[\nabla,\mathcal{D}]^0{}_{3\,1} \to \mu$, $A[\nabla,\mathcal{D}]^1{}_{4\,0} \to -\rho$, $A[\nabla,\mathcal{D}]^1{}_{4\,1} \to -\alpha$, $A[\nabla,\mathcal{D}]^0{}_{4\,0} \to \alpha$, $A[\nabla,\mathcal{D}]^0{}_{4\,1} \to \lambda \}$

We need the complex conjugate of all these relations.

In[139]:= `Map[Dagger, %, {2}]`

Out[139]= $\{A[\nabla,\mathcal{D}]\dagger^{1'}{}_{1\,0'} \to -\overline{\kappa}$, $A[\nabla,\mathcal{D}]\dagger^{1'}{}_{1\,1'} \to -\overline{\epsilon}$, $A[\nabla,\mathcal{D}]\dagger^{0'}{}_{1\,0'} \to \overline{\epsilon}$, $A[\nabla,\mathcal{D}]\dagger^{0'}{}_{1\,1'} \to \overline{\pi}$,

$A[\nabla,\mathcal{D}]\dagger^{1'}{}_{2\,0'} \to -\overline{\tau}$, $A[\nabla,\mathcal{D}]\dagger^{1'}{}_{2\,1'} \to -\overline{\gamma}$, $A[\nabla,\mathcal{D}]\dagger^{0'}{}_{2\,0'} \to \overline{\gamma}$, $A[\nabla,\mathcal{D}]\dagger^{0'}{}_{2\,1'} \to \overline{\nu}$,

$A[\nabla,\mathcal{D}]\dagger^{1'}{}_{4\,0'} \to -\overline{\sigma}$, $A[\nabla,\mathcal{D}]\dagger^{1'}{}_{4\,1'} \to -\overline{\beta}$, $A[\nabla,\mathcal{D}]\dagger^{0'}{}_{4\,0'} \to \overline{\beta}$, $A[\nabla,\mathcal{D}]\dagger^{0'}{}_{4\,1'} \to \overline{\mu}$,

$A[\nabla,\mathcal{D}]\dagger^{1'}{}_{3\,0'} \to -\overline{\rho}$, $A[\nabla,\mathcal{D}]\dagger^{1'}{}_{3\,1'} \to -\overline{\alpha}$, $A[\nabla,\mathcal{D}]\dagger^{0'}{}_{3\,0'} \to \overline{\alpha}$, $A[\nabla,\mathcal{D}]\dagger^{0'}{}_{3\,1'} \to \overline{\lambda} \}$

In[140]:= `% /. Rule -> ComponentValue`

Out[140]= $\{A[\nabla,\mathcal{D}]\dagger^{1'}{}_{1\,0'} \to -\overline{\kappa}$, $A[\nabla,\mathcal{D}]\dagger^{1'}{}_{1\,1'} \to -\overline{\epsilon}$, $A[\nabla,\mathcal{D}]\dagger^{0'}{}_{1\,0'} \to \overline{\epsilon}$, $A[\nabla,\mathcal{D}]\dagger^{0'}{}_{1\,1'} \to \overline{\pi}$,

$A[\nabla,\mathcal{D}]\dagger^{1'}{}_{2\,0'} \to -\overline{\tau}$, $A[\nabla,\mathcal{D}]\dagger^{1'}{}_{2\,1'} \to -\overline{\gamma}$, $A[\nabla,\mathcal{D}]\dagger^{0'}{}_{2\,0'} \to \overline{\gamma}$, $A[\nabla,\mathcal{D}]\dagger^{0'}{}_{2\,1'} \to \overline{\nu}$,

$A[\nabla,\mathcal{D}]\dagger^{1'}{}_{4\,0'} \to -\overline{\sigma}$, $A[\nabla,\mathcal{D}]\dagger^{1'}{}_{4\,1'} \to -\overline{\beta}$, $A[\nabla,\mathcal{D}]\dagger^{0'}{}_{4\,0'} \to \overline{\beta}$, $A[\nabla,\mathcal{D}]\dagger^{0'}{}_{4\,1'} \to \overline{\mu}$,

$A[\nabla,\mathcal{D}]\dagger^{1'}{}_{3\,0'} \to -\overline{\rho}$, $A[\nabla,\mathcal{D}]\dagger^{1'}{}_{3\,1'} \to -\overline{\alpha}$, $A[\nabla,\mathcal{D}]\dagger^{0'}{}_{3\,0'} \to \overline{\alpha}$, $A[\nabla,\mathcal{D}]\dagger^{0'}{}_{3\,1'} \to \overline{\lambda} \}$

Automatic replacement of the components of $A[\nabla,\mathcal{D}]^A{}_{bC}$ and $A[\nabla,\mathcal{D}]\dagger^{A'}{}_{bC'}$.

In[141]:= `RuleToSet@AChristoffelCDePDDyad;`

In[142]:= `RuleToSet@AChristoffelCDePDDyad†;`

■ **Components of the Levi-Civita connection in the null tetrad**

We need to obtain the components of the Levi-Civita connection (Ricci rotation coefficients) in the null tetrad. This is done by taking the covariant derivative of the set of equations relating the null tetrad elements and the spin dyad elements and using the expressions for the derivatives of the latter in terms of the Newman-Penrose spin coefficients.

As we know the relation between the spinor dyad and the null tetrad is

In[143]:= `SpinDyadToNullTetrad`

Out[143]= $\{o^A \overline{o}^{A'} \sigma^a{}_{AA'} == l^a$, $\iota^A \overline{\iota}^{A'} \sigma^a{}_{AA'} == n^a$, $o^A \overline{\iota}^{A'} \sigma^a{}_{AA'} == m^a$, $\overline{o}^{A'} \iota^A \sigma^a{}_{AA'} == \overline{m}^a\}$

We take now the derivative of each of these equations and expand them in components.

In[144]:= `Map[CDe[-b], %, {2}]`

Out[144]= $\{\overline{o}^{A'} \sigma^a{}_{AA'} (\nabla_b o^A) + o^A \sigma^a{}_{AA'} (\nabla_b \overline{o}^{A'}) == \nabla_b l^a$, $\overline{\iota}^{A'} \sigma^a{}_{AA'} (\nabla_b \iota^A) + \iota^A \sigma^a{}_{AA'} (\nabla_b \overline{\iota}^{A'}) == \nabla_b n^a$,

$\overline{\iota}^{A'} \sigma^a{}_{AA'} (\nabla_b o^A) + o^A \sigma^a{}_{AA'} (\nabla_b \overline{\iota}^{A'}) == \nabla_b m^a$, $\iota^A \sigma^a{}_{AA'} (\nabla_b \overline{o}^{A'}) + \overline{o}^{A'} \sigma^a{}_{AA'} (\nabla_b \iota^A) == \nabla_b \overline{m}^a\}$

In[145]:= **Map[ToBasis[NP], %, {2}]**

Out[145]= $\left\{ \overline{o}^{A'} \, \sigma^a_{AA'} \, \left(\nabla_b o^A \right) + o^A \, \sigma^a_{AA'} \, \left(\nabla_b \overline{o}^{A'} \right) \, == \, e_c{}^a \left(\nabla_b l^c \right), \, \overline{\iota}^{A'} \, \sigma^a_{AA'} \, \left(\nabla_b \iota^A \right) + \iota^A \, \sigma^a_{AA'} \, \left(\nabla_b \overline{\iota}^{A'} \right) \, == \, e_c{}^a \left(\nabla_b n^c \right), \right.$

$\left. \overline{\iota}^{A'} \, \sigma^a_{AA'} \, \left(\nabla_b o^A \right) + o^A \, \sigma^a_{AA'} \, \left(\nabla_b \overline{\iota}^{A'} \right) \, == \, e_c{}^a \left(\nabla_b m^c \right), \, \iota^A \, \sigma^a_{AA'} \, \left(\nabla_b \overline{o}^{A'} \right) + \overline{o}^{A'} \, \sigma^a_{AA'} \, \left(\nabla_b \iota^A \right) \, == \, e_c{}^a \left(\nabla_b \overline{m}^c \right) \right\}$

In[146]:= **Map[ToBasis[NP], %, {2}]**

** DefTensor: Defining tensor ChristoffelCDePDNP[a, -b, -c].

Out[146]= $\left\{ \overline{o}^{A'} \, \sigma^a_{AA'} \, \left(\nabla_b o^A \right) + o^A \, \sigma^a_{AA'} \, \left(\nabla_b \overline{o}^{A'} \right) \, == \, \Gamma[\nabla, \mathcal{D}]^a{}_{bc} \, l^c + \nabla_b l^a, \right.$

$\overline{\iota}^{A'} \, \sigma^a_{AA'} \, \left(\nabla_b \iota^A \right) + \iota^A \, \sigma^a_{AA'} \, \left(\nabla_b \overline{\iota}^{A'} \right) \, == \, \Gamma[\nabla, \mathcal{D}]^a{}_{bc} \, n^c + \nabla_b n^a,$

$\overline{\iota}^{A'} \, \sigma^a_{AA'} \, \left(\nabla_b o^A \right) + o^A \, \sigma^a_{AA'} \, \left(\nabla_b \overline{\iota}^{A'} \right) \, == \, \Gamma[\nabla, \mathcal{D}]^a{}_{bc} \, m^c + \nabla_b m^a,$

$\left. \iota^A \, \sigma^a_{AA'} \, \left(\nabla_b \overline{o}^{A'} \right) + \overline{o}^{A'} \, \sigma^a_{AA'} \, \left(\nabla_b \iota^A \right) \, == \, \Gamma[\nabla, \mathcal{D}]^a{}_{bc} \, \overline{m}^c + \nabla_b \overline{m}^a \right\}$

In[147]:= **Map[ToBasis[Dyad], %, {2}]**

Out[147]= $\left\{ A[\nabla, \mathcal{D}]^A{}_{bB} \, o^B \, \overline{o}^{A'} \, \sigma^a_{AA'} + \overline{o}^{A'} \, \sigma^a_{AA'} \, \left(\nabla_b o^A \right) + o^A \, \sigma^a_{AA'} \, \left(\nabla_b \overline{o}^{A'} \right) \, == \, \Gamma[\nabla, \mathcal{D}]^a{}_{bc} \, l^c + \nabla_b l^a, \right.$

$A[\nabla, \mathcal{D}]^A{}_{bB} \, \iota^B \, \overline{\iota}^{A'} \, \sigma^a_{AA'} + \overline{\iota}^{A'} \, \sigma^a_{AA'} \, \left(\nabla_b \iota^A \right) + \iota^A \, \sigma^a_{AA'} \, \left(\nabla_b \overline{\iota}^{A'} \right) \, == \, \Gamma[\nabla, \mathcal{D}]^a{}_{bc} \, n^c + \nabla_b n^a,$

$A[\nabla, \mathcal{D}]^A{}_{bB} \, o^B \, \overline{\iota}^{A'} \, \sigma^a_{AA'} + \overline{\iota}^{A'} \, \sigma^a_{AA'} \, \left(\nabla_b o^A \right) + o^A \, \sigma^a_{AA'} \, \left(\nabla_b \overline{\iota}^{A'} \right) \, == \, \Gamma[\nabla, \mathcal{D}]^a{}_{bc} \, m^c + \nabla_b m^a,$

$\left. A[\nabla, \mathcal{D}]^A{}_{bB} \, \overline{o}^{A'} \, \iota^B \, \sigma^a_{AA'} + \iota^A \, \sigma^a_{AA'} \, \left(\nabla_b \overline{o}^{A'} \right) + \overline{o}^{A'} \, \sigma^a_{AA'} \, \left(\nabla_b \iota^A \right) \, == \, \Gamma[\nabla, \mathcal{D}]^a{}_{bc} \, \overline{m}^c + \nabla_b \overline{m}^a \right\}$

In[148]:= **Map[ToBasis[Dyad†], %, {2}]**

Out[148]= $\left\{ A[\nabla, \mathcal{D}]^A{}_{bB} \, o^B \, \overline{o}^{A'} \, \sigma^a_{AA'} + A[\nabla, \mathcal{D}]\, {\dagger}^{A'}{}_{bB'} \, o^A \, \overline{o}^{B'} \, \sigma^a_{AA'} + \overline{o}^{A'} \, \sigma^a_{AA'} \, \left(\nabla_b o^A \right) + o^A \, \sigma^a_{AA'} \, \left(\nabla_b \overline{o}^{A'} \right) \, == \right.$

$\Gamma[\nabla, \mathcal{D}]^a{}_{bc} \, l^c + \nabla_b l^a, \, A[\nabla, \mathcal{D}]^A{}_{bB} \, \iota^B \, \overline{\iota}^{A'} \, \sigma^a_{AA'} + A[\nabla, \mathcal{D}]\, {\dagger}^{A'}{}_{bB'} \, \iota^A \, \overline{\iota}^{B'} \, \sigma^a_{AA'} +$

$\overline{\iota}^{A'} \, \sigma^a_{AA'} \, \left(\nabla_b \iota^A \right) + \iota^A \, \sigma^a_{AA'} \, \left(\nabla_b \overline{\iota}^{A'} \right) \, == \, \Gamma[\nabla, \mathcal{D}]^a{}_{bc} \, n^c + \nabla_b n^a,$

$A[\nabla, \mathcal{D}]^A{}_{bB} \, o^B \, \overline{\iota}^{A'} \, \sigma^a_{AA'} + A[\nabla, \mathcal{D}]\, {\dagger}^{A'}{}_{bB'} \, o^A \, \overline{\iota}^{B'} \, \sigma^a_{AA'} + \overline{\iota}^{A'} \, \sigma^a_{AA'} \, \left(\nabla_b o^A \right) + o^A \, \sigma^a_{AA'} \, \left(\nabla_b \overline{\iota}^{A'} \right) \, == \,$

$\Gamma[\nabla, \mathcal{D}]^a{}_{bc} \, m^c + \nabla_b m^a, \, A[\nabla, \mathcal{D}]\, {\dagger}^{A'}{}_{bB'} \, \overline{o}^{B'} \, \iota^A \, \sigma^a_{AA'} + A[\nabla, \mathcal{D}]^A{}_{bB} \, \overline{o}^{A'} \, \iota^B \, \sigma^a_{AA'} +$

$\left. \iota^A \, \sigma^a_{AA'} \, \left(\nabla_b \overline{o}^{A'} \right) + \overline{o}^{A'} \, \sigma^a_{AA'} \, \left(\nabla_b \iota^A \right) \, == \, \Gamma[\nabla, \mathcal{D}]^a{}_{bc} \, \overline{m}^c + \nabla_b \overline{m}^a \right\}$

All the ocurrences of the spin covariant derivative are now acting on scalar functions. Therefore we may replace them by any other covariant derivative, in particular PDNP.

In[149]:= **% /. CDe → PDNP**

Out[149]= $\left\{ A[\nabla, \mathcal{D}]^A{}_{bB} \, o^B \, \overline{o}^{A'} \, \sigma^a_{AA'} + A[\nabla, \mathcal{D}]\, {\dagger}^{A'}{}_{bB'} \, o^A \, \overline{o}^{B'} \, \sigma^a_{AA'} + \overline{o}^{A'} \, \sigma^a_{AA'} \, \left(\mathcal{D}_b o^A \right) + o^A \, \sigma^a_{AA'} \, \left(\mathcal{D}_b \overline{o}^{A'} \right) \, == \right.$

$\Gamma[\nabla, \mathcal{D}]^a{}_{bc} \, l^c + \mathcal{D}_b l^a, \, A[\nabla, \mathcal{D}]^A{}_{bB} \, \iota^B \, \overline{\iota}^{A'} \, \sigma^a_{AA'} + A[\nabla, \mathcal{D}]\, {\dagger}^{A'}{}_{bB'} \, \iota^A \, \overline{\iota}^{B'} \, \sigma^a_{AA'} +$

$\overline{\iota}^{A'} \, \sigma^a_{AA'} \, \left(\mathcal{D}_b \iota^A \right) + \iota^A \, \sigma^a_{AA'} \, \left(\mathcal{D}_b \overline{\iota}^{A'} \right) \, == \, \Gamma[\nabla, \mathcal{D}]^a{}_{bc} \, n^c + \mathcal{D}_b n^a,$

$A[\nabla, \mathcal{D}]^A{}_{bB} \, o^B \, \overline{\iota}^{A'} \, \sigma^a_{AA'} + A[\nabla, \mathcal{D}]\, {\dagger}^{A'}{}_{bB'} \, o^A \, \overline{\iota}^{B'} \, \sigma^a_{AA'} + \overline{\iota}^{A'} \, \sigma^a_{AA'} \, \left(\mathcal{D}_b o^A \right) + o^A \, \sigma^a_{AA'} \, \left(\mathcal{D}_b \overline{\iota}^{A'} \right) \, == \,$

$\Gamma[\nabla, \mathcal{D}]^a{}_{bc} \, m^c + \mathcal{D}_b m^a, \, A[\nabla, \mathcal{D}]\, {\dagger}^{A'}{}_{bB'} \, \overline{o}^{B'} \, \iota^A \, \sigma^a_{AA'} + A[\nabla, \mathcal{D}]^A{}_{bB} \, \overline{o}^{A'} \, \iota^B \, \sigma^a_{AA'} +$

$\left. \iota^A \, \sigma^a_{AA'} \, \left(\mathcal{D}_b \overline{o}^{A'} \right) + \overline{o}^{A'} \, \sigma^a_{AA'} \, \left(\mathcal{D}_b \iota^A \right) \, == \, \Gamma[\nabla, \mathcal{D}]^a{}_{bc} \, \overline{m}^c + \mathcal{D}_b \overline{m}^a \right\}$

In[150]:= **Map[TraceBasisDummy, %, {2}];**

In[151]:= **Flatten@ComponentArray@%**

Out[151]= $\{ \epsilon + \overline{\epsilon} == \Gamma[\nabla,\mathcal{D}]^1{}_{1\,1}, \gamma + \overline{\gamma} == \Gamma[\nabla,\mathcal{D}]^1{}_{2\,1}, \overline{\alpha} + \beta == \Gamma[\nabla,\mathcal{D}]^1{}_{3\,1}, \alpha + \overline{\beta} == \Gamma[\nabla,\mathcal{D}]^1{}_{4\,1}, 0 == \Gamma[\nabla,\mathcal{D}]^2{}_{1\,1},$

$0 == \Gamma[\nabla,\mathcal{D}]^2{}_{2\,1}, 0 == \Gamma[\nabla,\mathcal{D}]^2{}_{3\,1}, 0 == \Gamma[\nabla,\mathcal{D}]^2{}_{4\,1}, -\overline{\kappa} == \Gamma[\nabla,\mathcal{D}]^3{}_{1\,1}, -\overline{\tau} == \Gamma[\nabla,\mathcal{D}]^3{}_{2\,1}, -\overline{\rho} == \Gamma[\nabla,\mathcal{D}]^3{}_{3\,1},$

$-\overline{\sigma} == \Gamma[\nabla,\mathcal{D}]^3{}_{4\,1}, -\kappa == \Gamma[\nabla,\mathcal{D}]^4{}_{1\,1}, -\tau == \Gamma[\nabla,\mathcal{D}]^4{}_{2\,1}, -\sigma == \Gamma[\nabla,\mathcal{D}]^4{}_{3\,1}, -\rho == \Gamma[\nabla,\mathcal{D}]^4{}_{4\,1},$

$0 == \Gamma[\nabla,\mathcal{D}]^1{}_{1\,2}, 0 == \Gamma[\nabla,\mathcal{D}]^1{}_{2\,2}, 0 == \Gamma[\nabla,\mathcal{D}]^1{}_{3\,2}, 0 == \Gamma[\nabla,\mathcal{D}]^1{}_{4\,2}, -\epsilon - \overline{\epsilon} == \Gamma[\nabla,\mathcal{D}]^2{}_{1\,2},$

$-\gamma - \overline{\gamma} == \Gamma[\nabla,\mathcal{D}]^2{}_{2\,2}, -\overline{\alpha} - \beta == \Gamma[\nabla,\mathcal{D}]^2{}_{3\,2}, -\alpha - \overline{\beta} == \Gamma[\nabla,\mathcal{D}]^2{}_{4\,2}, \pi == \Gamma[\nabla,\mathcal{D}]^3{}_{1\,2},$

$\nu == \Gamma[\nabla,\mathcal{D}]^3{}_{2\,2}, \mu == \Gamma[\nabla,\mathcal{D}]^3{}_{3\,2}, \lambda == \Gamma[\nabla,\mathcal{D}]^3{}_{4\,2}, \overline{\pi} == \Gamma[\nabla,\mathcal{D}]^4{}_{1\,2}, \overline{\nu} == \Gamma[\nabla,\mathcal{D}]^4{}_{2\,2},$

$\overline{\lambda} == \Gamma[\nabla,\mathcal{D}]^4{}_{3\,2}, \overline{\mu} == \Gamma[\nabla,\mathcal{D}]^4{}_{4\,2}, \overline{\pi} == \Gamma[\nabla,\mathcal{D}]^1{}_{1\,3}, \overline{\nu} == \Gamma[\nabla,\mathcal{D}]^1{}_{2\,3}, \overline{\lambda} == \Gamma[\nabla,\mathcal{D}]^1{}_{3\,3},$

$\overline{\mu} == \Gamma[\nabla,\mathcal{D}]^1{}_{4\,3}, -\kappa == \Gamma[\nabla,\mathcal{D}]^2{}_{1\,3}, -\tau == \Gamma[\nabla,\mathcal{D}]^2{}_{2\,3}, -\sigma == \Gamma[\nabla,\mathcal{D}]^2{}_{3\,3}, -\rho == \Gamma[\nabla,\mathcal{D}]^2{}_{4\,3},$

$\epsilon - \overline{\epsilon} == \Gamma[\nabla,\mathcal{D}]^3{}_{1\,3}, \gamma - \overline{\gamma} == \Gamma[\nabla,\mathcal{D}]^3{}_{2\,3}, -\overline{\alpha} + \beta == \Gamma[\nabla,\mathcal{D}]^3{}_{3\,3}, \alpha - \overline{\beta} == \Gamma[\nabla,\mathcal{D}]^3{}_{4\,3},$

$0 == \Gamma[\nabla,\mathcal{D}]^4{}_{1\,3}, 0 == \Gamma[\nabla,\mathcal{D}]^4{}_{2\,3}, 0 == \Gamma[\nabla,\mathcal{D}]^4{}_{3\,3}, 0 == \Gamma[\nabla,\mathcal{D}]^4{}_{4\,3}, \pi == \Gamma[\nabla,\mathcal{D}]^1{}_{1\,4}, \nu == \Gamma[\nabla,\mathcal{D}]^1{}_{2\,4},$

$\mu == \Gamma[\nabla,\mathcal{D}]^1{}_{3\,4}, \lambda == \Gamma[\nabla,\mathcal{D}]^1{}_{4\,4}, -\overline{\kappa} == \Gamma[\nabla,\mathcal{D}]^2{}_{1\,4}, -\overline{\tau} == \Gamma[\nabla,\mathcal{D}]^2{}_{2\,4}, -\overline{\rho} == \Gamma[\nabla,\mathcal{D}]^2{}_{3\,4},$

$-\overline{\sigma} == \Gamma[\nabla,\mathcal{D}]^2{}_{4\,4}, 0 == \Gamma[\nabla,\mathcal{D}]^3{}_{1\,4}, 0 == \Gamma[\nabla,\mathcal{D}]^3{}_{2\,4}, 0 == \Gamma[\nabla,\mathcal{D}]^3{}_{3\,4}, 0 == \Gamma[\nabla,\mathcal{D}]^3{}_{4\,4},$

$-\epsilon + \overline{\epsilon} == \Gamma[\nabla,\mathcal{D}]^4{}_{1\,4}, -\gamma + \overline{\gamma} == \Gamma[\nabla,\mathcal{D}]^4{}_{2\,4}, \overline{\alpha} - \beta == \Gamma[\nabla,\mathcal{D}]^4{}_{3\,4}, -\alpha + \overline{\beta} == \Gamma[\nabla,\mathcal{D}]^4{}_{4\,4} \}$

In[152]:= **Solve[%, Union@Cases[%, _ChristoffelCDePDNP, Infinity]] // Flatten // Simplify**

Out[152]= $\{ \Gamma[\nabla,\mathcal{D}]^1{}_{1\,1} \to \epsilon + \overline{\epsilon}, \Gamma[\nabla,\mathcal{D}]^1{}_{2\,1} \to \gamma + \overline{\gamma}, \Gamma[\nabla,\mathcal{D}]^1{}_{3\,1} \to \overline{\alpha} + \beta, \Gamma[\nabla,\mathcal{D}]^1{}_{4\,1} \to \alpha + \overline{\beta}, \Gamma[\nabla,\mathcal{D}]^2{}_{1\,1} \to 0,$

$\Gamma[\nabla,\mathcal{D}]^2{}_{2\,1} \to 0, \Gamma[\nabla,\mathcal{D}]^2{}_{3\,1} \to 0, \Gamma[\nabla,\mathcal{D}]^2{}_{4\,1} \to 0, \Gamma[\nabla,\mathcal{D}]^3{}_{1\,1} \to -\overline{\kappa}, \Gamma[\nabla,\mathcal{D}]^3{}_{2\,1} \to -\overline{\tau}, \Gamma[\nabla,\mathcal{D}]^3{}_{3\,1} \to -\overline{\rho},$

$\Gamma[\nabla,\mathcal{D}]^3{}_{4\,1} \to -\overline{\sigma}, \Gamma[\nabla,\mathcal{D}]^4{}_{1\,1} \to -\kappa, \Gamma[\nabla,\mathcal{D}]^4{}_{2\,1} \to -\tau, \Gamma[\nabla,\mathcal{D}]^4{}_{3\,1} \to -\sigma, \Gamma[\nabla,\mathcal{D}]^4{}_{4\,1} \to -\rho,$

$\Gamma[\nabla,\mathcal{D}]^1{}_{1\,2} \to 0, \Gamma[\nabla,\mathcal{D}]^1{}_{2\,2} \to 0, \Gamma[\nabla,\mathcal{D}]^1{}_{3\,2} \to 0, \Gamma[\nabla,\mathcal{D}]^1{}_{4\,2} \to 0, \Gamma[\nabla,\mathcal{D}]^2{}_{1\,2} \to -\epsilon - \overline{\epsilon},$

$\Gamma[\nabla,\mathcal{D}]^2{}_{2\,2} \to -\gamma - \overline{\gamma}, \Gamma[\nabla,\mathcal{D}]^2{}_{3\,2} \to -\overline{\alpha} - \beta, \Gamma[\nabla,\mathcal{D}]^2{}_{4\,2} \to -\alpha - \overline{\beta}, \Gamma[\nabla,\mathcal{D}]^3{}_{1\,2} \to \pi, \Gamma[\nabla,\mathcal{D}]^3{}_{2\,2} \to \nu,$

$\Gamma[\nabla,\mathcal{D}]^3{}_{3\,2} \to \mu, \Gamma[\nabla,\mathcal{D}]^3{}_{4\,2} \to \lambda, \Gamma[\nabla,\mathcal{D}]^4{}_{1\,2} \to \overline{\pi}, \Gamma[\nabla,\mathcal{D}]^4{}_{2\,2} \to \overline{\nu}, \Gamma[\nabla,\mathcal{D}]^4{}_{3\,2} \to \overline{\lambda}, \Gamma[\nabla,\mathcal{D}]^4{}_{4\,2} \to \overline{\mu},$

$\Gamma[\nabla,\mathcal{D}]^1{}_{1\,3} \to \overline{\pi}, \Gamma[\nabla,\mathcal{D}]^1{}_{2\,3} \to \overline{\nu}, \Gamma[\nabla,\mathcal{D}]^1{}_{3\,3} \to \overline{\lambda}, \Gamma[\nabla,\mathcal{D}]^1{}_{4\,3} \to \overline{\mu}, \Gamma[\nabla,\mathcal{D}]^2{}_{1\,3} \to -\kappa, \Gamma[\nabla,\mathcal{D}]^2{}_{2\,3} \to -\tau,$

$\Gamma[\nabla,\mathcal{D}]^2{}_{3\,3} \to -\sigma, \Gamma[\nabla,\mathcal{D}]^2{}_{4\,3} \to -\rho, \Gamma[\nabla,\mathcal{D}]^3{}_{1\,3} \to \epsilon - \overline{\epsilon}, \Gamma[\nabla,\mathcal{D}]^3{}_{2\,3} \to \gamma - \overline{\gamma}, \Gamma[\nabla,\mathcal{D}]^3{}_{3\,3} \to -\overline{\alpha} + \beta,$

$\Gamma[\nabla,\mathcal{D}]^3{}_{4\,3} \to \alpha - \overline{\beta}, \Gamma[\nabla,\mathcal{D}]^4{}_{1\,3} \to 0, \Gamma[\nabla,\mathcal{D}]^4{}_{2\,3} \to 0, \Gamma[\nabla,\mathcal{D}]^4{}_{3\,3} \to 0, \Gamma[\nabla,\mathcal{D}]^4{}_{4\,3} \to 0, \Gamma[\nabla,\mathcal{D}]^1{}_{1\,4} \to \pi,$

$\Gamma[\nabla,\mathcal{D}]^1{}_{2\,4} \to \nu, \Gamma[\nabla,\mathcal{D}]^1{}_{3\,4} \to \mu, \Gamma[\nabla,\mathcal{D}]^1{}_{4\,4} \to \lambda, \Gamma[\nabla,\mathcal{D}]^2{}_{1\,4} \to -\overline{\kappa}, \Gamma[\nabla,\mathcal{D}]^2{}_{2\,4} \to -\overline{\tau},$

$\Gamma[\nabla,\mathcal{D}]^2{}_{3\,4} \to -\overline{\rho}, \Gamma[\nabla,\mathcal{D}]^2{}_{4\,4} \to -\overline{\sigma}, \Gamma[\nabla,\mathcal{D}]^3{}_{1\,4} \to 0, \Gamma[\nabla,\mathcal{D}]^3{}_{2\,4} \to 0, \Gamma[\nabla,\mathcal{D}]^3{}_{3\,4} \to 0, \Gamma[\nabla,\mathcal{D}]^3{}_{4\,4} \to 0,$

$\Gamma[\nabla,\mathcal{D}]^4{}_{1\,4} \to -\epsilon + \overline{\epsilon}, \Gamma[\nabla,\mathcal{D}]^4{}_{2\,4} \to -\gamma + \overline{\gamma}, \Gamma[\nabla,\mathcal{D}]^4{}_{3\,4} \to \overline{\alpha} - \beta, \Gamma[\nabla,\mathcal{D}]^4{}_{4\,4} \to -\alpha + \overline{\beta} \}$

These are the components of the Levi-Civita connection in the null tetrad written in terms of the spin coefficients. We store them.

In[153]:= **% /. Rule → ComponentValue;**

We show the list of Ricci rotation coefficients.

In[154]:= **TensorValues@ChristoffelCDePDNP**

Out[154]= $\text{FoldedRule}\Big[\{\}, \Big\{\Gamma[\nabla,\mathcal{D}]^1{}_{11} \to \epsilon + \overline{\epsilon}, \ \Gamma[\nabla,\mathcal{D}]^1{}_{21} \to \gamma + \overline{\gamma}, \ \Gamma[\nabla,\mathcal{D}]^1{}_{31} \to \overline{\alpha} + \beta, \ \Gamma[\nabla,\mathcal{D}]^1{}_{41} \to \alpha + \overline{\beta},$

$\Gamma[\nabla,\mathcal{D}]^2{}_{11} \to 0, \ \Gamma[\nabla,\mathcal{D}]^2{}_{21} \to 0, \ \Gamma[\nabla,\mathcal{D}]^2{}_{31} \to 0, \ \Gamma[\nabla,\mathcal{D}]^2{}_{41} \to 0, \ \Gamma[\nabla,\mathcal{D}]^3{}_{11} \to -\overline{\kappa}, \ \Gamma[\nabla,\mathcal{D}]^3{}_{21} \to -\overline{\tau},$

$\Gamma[\nabla,\mathcal{D}]^3{}_{31} \to -\overline{\rho}, \ \Gamma[\nabla,\mathcal{D}]^3{}_{41} \to -\overline{\sigma}, \ \Gamma[\nabla,\mathcal{D}]^4{}_{11} \to -\kappa, \ \Gamma[\nabla,\mathcal{D}]^4{}_{21} \to -\tau, \ \Gamma[\nabla,\mathcal{D}]^4{}_{31} \to -\sigma,$

$\Gamma[\nabla,\mathcal{D}]^4{}_{41} \to -\rho, \ \Gamma[\nabla,\mathcal{D}]^1{}_{12} \to 0, \ \Gamma[\nabla,\mathcal{D}]^1{}_{22} \to 0, \ \Gamma[\nabla,\mathcal{D}]^1{}_{32} \to 0, \ \Gamma[\nabla,\mathcal{D}]^1{}_{42} \to 0,$

$\Gamma[\nabla,\mathcal{D}]^2{}_{12} \to -\epsilon - \overline{\epsilon}, \ \Gamma[\nabla,\mathcal{D}]^2{}_{22} \to -\gamma - \overline{\gamma}, \ \Gamma[\nabla,\mathcal{D}]^2{}_{32} \to -\overline{\alpha} - \beta, \ \Gamma[\nabla,\mathcal{D}]^2{}_{42} \to -\alpha - \overline{\beta},$

$\Gamma[\nabla,\mathcal{D}]^3{}_{12} \to \pi, \ \Gamma[\nabla,\mathcal{D}]^3{}_{22} \to \nu, \ \Gamma[\nabla,\mathcal{D}]^3{}_{32} \to \mu, \ \Gamma[\nabla,\mathcal{D}]^3{}_{42} \to \lambda, \ \Gamma[\nabla,\mathcal{D}]^4{}_{12} \to \overline{\pi}, \ \Gamma[\nabla,\mathcal{D}]^4{}_{22} \to \overline{\nu},$

$\Gamma[\nabla,\mathcal{D}]^4{}_{32} \to \overline{\lambda}, \ \Gamma[\nabla,\mathcal{D}]^4{}_{42} \to \overline{\mu}, \ \Gamma[\nabla,\mathcal{D}]^1{}_{13} \to \overline{\pi}, \ \Gamma[\nabla,\mathcal{D}]^1{}_{23} \to \overline{\nu}, \ \Gamma[\nabla,\mathcal{D}]^1{}_{33} \to \overline{\lambda},$

$\Gamma[\nabla,\mathcal{D}]^1{}_{43} \to \overline{\mu}, \ \Gamma[\nabla,\mathcal{D}]^2{}_{13} \to -\kappa, \ \Gamma[\nabla,\mathcal{D}]^2{}_{23} \to -\tau, \ \Gamma[\nabla,\mathcal{D}]^2{}_{33} \to -\sigma, \ \Gamma[\nabla,\mathcal{D}]^2{}_{43} \to -\rho,$

$\Gamma[\nabla,\mathcal{D}]^3{}_{13} \to \epsilon - \overline{\epsilon}, \ \Gamma[\nabla,\mathcal{D}]^3{}_{23} \to \gamma - \overline{\gamma}, \ \Gamma[\nabla,\mathcal{D}]^3{}_{33} \to -\overline{\alpha} + \beta, \ \Gamma[\nabla,\mathcal{D}]^3{}_{43} \to \alpha - \overline{\beta},$

$\Gamma[\nabla,\mathcal{D}]^4{}_{13} \to 0, \ \Gamma[\nabla,\mathcal{D}]^4{}_{23} \to 0, \ \Gamma[\nabla,\mathcal{D}]^4{}_{33} \to 0, \ \Gamma[\nabla,\mathcal{D}]^4{}_{43} \to 0, \ \Gamma[\nabla,\mathcal{D}]^1{}_{14} \to \pi, \ \Gamma[\nabla,\mathcal{D}]^1{}_{24} \to \nu,$

$\Gamma[\nabla,\mathcal{D}]^1{}_{34} \to \mu, \ \Gamma[\nabla,\mathcal{D}]^1{}_{44} \to \lambda, \ \Gamma[\nabla,\mathcal{D}]^2{}_{14} \to -\overline{\kappa}, \ \Gamma[\nabla,\mathcal{D}]^2{}_{24} \to -\overline{\tau}, \ \Gamma[\nabla,\mathcal{D}]^2{}_{34} \to -\overline{\rho},$

$\Gamma[\nabla,\mathcal{D}]^2{}_{44} \to -\overline{\sigma}, \ \Gamma[\nabla,\mathcal{D}]^3{}_{14} \to 0, \ \Gamma[\nabla,\mathcal{D}]^3{}_{24} \to 0, \ \Gamma[\nabla,\mathcal{D}]^3{}_{34} \to 0, \ \Gamma[\nabla,\mathcal{D}]^3{}_{44} \to 0,$

$\Gamma[\nabla,\mathcal{D}]^4{}_{14} \to -\epsilon + \overline{\epsilon}, \ \Gamma[\nabla,\mathcal{D}]^4{}_{24} \to -\gamma + \overline{\gamma}, \ \Gamma[\nabla,\mathcal{D}]^4{}_{34} \to \overline{\alpha} - \beta, \ \Gamma[\nabla,\mathcal{D}]^4{}_{44} \to -\alpha + \overline{\beta} \Big\}\Big]$

From now on, each Ricci rotation coefficient will be automatically replaced by its value in terms of the Newman-Penrose spin coefficients.

In[155]:= **RuleToSet@ChristoffelCDePDNP;**

■ **Commutation relations**

In this subsection we calculate the commutation coefficients of the elements of the null tetrad in terms of the spin coefficients. This is achieved by calculating the brackets of the elements of the null tetrad which in turn requires the commutation coefficients of the null tetrad elements. These coeffients are the components of the tensor `TorsionPDNP`.

The first step is the creation of a folded rule containing the independent components of `TorsionPDNP` in the null tetrad.

In[156]:= **AllComponentValues@ToBasis[NP]@TorsionPDNP[a, -b, -c]**

Out[156]= $\text{FoldedRule}\Big[\Big\{T^1{}_{11} \to 0, \ T^1{}_{21} \to -T^1{}_{12}, \ T^1{}_{22} \to 0, \ T^1{}_{31} \to -T^1{}_{13}, \ T^1{}_{32} \to -T^1{}_{23}, \ T^1{}_{33} \to 0,$

$T^1{}_{41} \to -T^1{}_{14}, \ T^1{}_{42} \to -T^1{}_{24}, \ T^1{}_{43} \to -T^1{}_{34}, \ T^1{}_{44} \to 0, \ T^2{}_{11} \to 0, \ T^2{}_{21} \to -T^2{}_{12}, \ T^2{}_{22} \to 0,$

$T^2{}_{31} \to -T^2{}_{13}, \ T^2{}_{32} \to -T^2{}_{23}, \ T^2{}_{33} \to 0, \ T^2{}_{41} \to -T^2{}_{14}, \ T^2{}_{42} \to -T^2{}_{24}, \ T^2{}_{43} \to -T^2{}_{34},$

$T^2{}_{44} \to 0, \ T^3{}_{11} \to 0, \ T^3{}_{21} \to -T^3{}_{12}, \ T^3{}_{22} \to 0, \ T^3{}_{31} \to -T^3{}_{13}, \ T^3{}_{32} \to -T^3{}_{23}, \ T^3{}_{33} \to 0,$

$T^3{}_{41} \to -T^3{}_{14}, \ T^3{}_{42} \to -T^3{}_{24}, \ T^3{}_{43} \to -T^3{}_{34}, \ T^3{}_{44} \to 0, \ T^4{}_{11} \to 0, \ T^4{}_{21} \to -T^4{}_{12}, \ T^4{}_{22} \to 0,$

$T^4{}_{31} \to -T^4{}_{13}, \ T^4{}_{32} \to -T^4{}_{23}, \ T^4{}_{33} \to 0, \ T^4{}_{41} \to -T^4{}_{14}, \ T^4{}_{42} \to -T^4{}_{24}, \ T^4{}_{43} \to -T^4{}_{34}, \ T^4{}_{44} \to 0\Big\},$

$\Big\{T^1{}_{12} \to T^1{}_{12}, \ T^1{}_{13} \to T^1{}_{13}, \ T^1{}_{14} \to T^1{}_{14}, \ T^1{}_{23} \to T^1{}_{23}, \ T^1{}_{24} \to T^1{}_{24}, \ T^1{}_{34} \to T^1{}_{34}, \ T^2{}_{12} \to T^2{}_{12}, \ T^2{}_{13} \to T^2{}_{13},$

$T^2{}_{14} \to T^2{}_{14}, \ T^2{}_{23} \to T^2{}_{23}, \ T^2{}_{24} \to T^2{}_{24}, \ T^2{}_{34} \to T^2{}_{34}, \ T^3{}_{12} \to T^3{}_{12}, \ T^3{}_{13} \to T^3{}_{13}, \ T^3{}_{14} \to T^3{}_{14}, \ T^3{}_{23} \to T^3{}_{23},$

$T^3{}_{24} \to T^3{}_{24}, \ T^3{}_{34} \to T^3{}_{34}, \ T^4{}_{12} \to T^4{}_{12}, \ T^4{}_{13} \to T^4{}_{13}, \ T^4{}_{14} \to T^4{}_{14}, \ T^4{}_{23} \to T^4{}_{23}, \ T^4{}_{24} \to T^4{}_{24}, \ T^4{}_{34} \to T^4{}_{34}\Big\}\Big]$

Next we compute the explicit expression for the bracket of each pair of elements of the null tetrad.

In[157]:= **Bracket[b][Basis[{-c, -NP}, a], Basis[{-d, -NP}, a]] ==**
ChangeTorsion[Bracket[b][Basis[{-c, -NP}, a], Basis[{-d, -NP}, a]], PDNP, CDe]

Out[157]= $-T^b{}_{cd} == \Gamma[\nabla,\mathcal{D}]^b{}_{cd} - \Gamma[\nabla,\mathcal{D}]^b{}_{dc}$

In[158]:= **ToBasis[NP] /@%**

Out[158]= $-T^b{}_{cd} == \Gamma[\nabla,\mathcal{D}]^b{}_{cd} - \Gamma[\nabla,\mathcal{D}]^b{}_{dc}$

Finally we compute and store the components of `TorsionPDNP` in the null tetrad.

In[159]:= **Flatten@ComponentArray@%**

Out[159]= $\{-T^1{}_{11} == 0, -T^1{}_{12} == -\gamma - \overline{\gamma}, -T^1{}_{13} == -\overline{\alpha} - \beta + \overline{\pi}, -T^1{}_{14} == -\alpha - \overline{\beta} + \pi, -T^1{}_{21} == \gamma + \overline{\gamma}, -T^1{}_{22} == 0,$
$-T^1{}_{23} == \overline{\nu}, -T^1{}_{24} == \nu, -T^1{}_{31} == \overline{\alpha} + \beta - \overline{\pi}, -T^1{}_{32} == -\overline{\nu}, -T^1{}_{33} == 0, -T^1{}_{34} == \mu - \overline{\mu},$
$-T^1{}_{41} == \alpha + \overline{\beta} - \pi, -T^1{}_{42} == -\nu, -T^1{}_{43} == -\mu + \overline{\mu}, -T^1{}_{44} == 0, -T^2{}_{11} == 0, -T^2{}_{12} == -\epsilon - \overline{\epsilon},$
$-T^2{}_{13} == -\kappa, -T^2{}_{14} == -\overline{\kappa}, -T^2{}_{21} == \epsilon + \overline{\epsilon}, -T^2{}_{22} == 0, -T^2{}_{23} == \overline{\alpha} + \beta - \tau, -T^2{}_{24} == \alpha + \overline{\beta} - \overline{\tau},$
$-T^2{}_{31} == \kappa, -T^2{}_{32} == -\overline{\alpha} - \beta + \tau, -T^2{}_{33} == 0, -T^2{}_{34} == \rho - \overline{\rho}, -T^2{}_{41} == \overline{\kappa}, -T^2{}_{42} == -\alpha - \overline{\beta} + \overline{\tau},$
$-T^2{}_{43} == -\rho + \overline{\rho}, -T^2{}_{44} == 0, -T^3{}_{11} == 0, -T^3{}_{12} == \pi + \overline{\tau}, -T^3{}_{13} == \epsilon - \overline{\epsilon} + \overline{\rho}, -T^3{}_{14} == \overline{\sigma},$
$-T^3{}_{21} == -\pi - \overline{\tau}, -T^3{}_{22} == 0, -T^3{}_{23} == \gamma - \overline{\gamma} - \mu, -T^3{}_{24} == -\lambda, -T^3{}_{31} == -\epsilon + \overline{\epsilon} - \overline{\rho},$
$-T^3{}_{32} == -\gamma + \overline{\gamma} + \mu, -T^3{}_{33} == 0, -T^3{}_{34} == -\alpha + \overline{\beta}, -T^3{}_{41} == -\overline{\sigma}, -T^3{}_{42} == \lambda, -T^3{}_{43} == \alpha - \overline{\beta},$
$-T^3{}_{44} == 0, -T^4{}_{11} == 0, -T^4{}_{12} == \overline{\pi} + \tau, -T^4{}_{13} == \sigma, -T^4{}_{14} == -\epsilon + \overline{\epsilon} + \rho, -T^4{}_{21} == -\overline{\pi} - \tau,$
$-T^4{}_{22} == 0, -T^4{}_{23} == -\overline{\lambda}, -T^4{}_{24} == -\gamma + \overline{\gamma} - \overline{\mu}, -T^4{}_{31} == -\sigma, -T^4{}_{32} == \overline{\lambda}, -T^4{}_{33} == 0,$
$-T^4{}_{34} == \overline{\alpha} - \beta, -T^4{}_{41} == \epsilon - \overline{\epsilon} - \rho, -T^4{}_{42} == \gamma - \overline{\gamma} + \overline{\mu}, -T^4{}_{43} == -\overline{\alpha} + \beta, -T^4{}_{44} == 0\}$

In[160]:= **Union@ToValues@%**

Out[160]= $\{\text{True}, -T^1{}_{12} == -\gamma - \overline{\gamma}, T^1{}_{12} == \gamma + \overline{\gamma}, -T^1{}_{13} == -\overline{\alpha} - \beta + \overline{\pi}, T^1{}_{13} == \overline{\alpha} + \beta - \overline{\pi},$
$-T^1{}_{14} == -\alpha - \overline{\beta} + \pi, T^1{}_{14} == \alpha + \overline{\beta} - \pi, -T^1{}_{23} == \overline{\nu}, T^1{}_{23} == -\overline{\nu}, -T^1{}_{24} == \nu, T^1{}_{24} == -\nu,$
$-T^1{}_{34} == \mu - \overline{\mu}, T^1{}_{34} == -\mu + \overline{\mu}, -T^2{}_{12} == -\epsilon - \overline{\epsilon}, T^2{}_{12} == \epsilon + \overline{\epsilon}, -T^2{}_{13} == -\kappa, T^2{}_{13} == \kappa,$
$-T^2{}_{14} == -\overline{\kappa}, T^2{}_{14} == \overline{\kappa}, -T^2{}_{23} == \overline{\alpha} + \beta - \tau, T^2{}_{23} == -\overline{\alpha} - \beta + \tau, -T^2{}_{24} == \alpha + \overline{\beta} - \overline{\tau},$
$T^2{}_{24} == -\alpha - \overline{\beta} + \overline{\tau}, -T^2{}_{34} == \rho - \overline{\rho}, T^2{}_{34} == -\rho + \overline{\rho}, -T^3{}_{12} == \pi + \overline{\tau}, T^3{}_{12} == -\pi - \overline{\tau},$
$-T^3{}_{13} == \epsilon - \overline{\epsilon} + \overline{\rho}, T^3{}_{13} == -\epsilon + \overline{\epsilon} - \overline{\rho}, -T^3{}_{14} == \overline{\sigma}, T^3{}_{14} == -\overline{\sigma}, -T^3{}_{23} == \gamma - \overline{\gamma} - \mu,$
$T^3{}_{23} == -\gamma + \overline{\gamma} + \mu, -T^3{}_{24} == -\lambda, T^3{}_{24} == \lambda, -T^3{}_{34} == -\alpha + \overline{\beta}, T^3{}_{34} == \alpha - \overline{\beta}, -T^4{}_{12} == \overline{\pi} + \tau,$
$T^4{}_{12} == -\overline{\pi} - \tau, -T^4{}_{13} == \sigma, T^4{}_{13} == -\sigma, -T^4{}_{14} == -\epsilon + \overline{\epsilon} + \rho, T^4{}_{14} == \epsilon - \overline{\epsilon} - \rho, -T^4{}_{23} == -\overline{\lambda},$
$T^4{}_{23} == \overline{\lambda}, -T^4{}_{24} == -\gamma + \overline{\gamma} - \overline{\mu}, T^4{}_{24} == \gamma - \overline{\gamma} + \overline{\mu}, -T^4{}_{34} == \overline{\alpha} - \beta, T^4{}_{34} == -\overline{\alpha} + \beta\}$

In[161]:= **Solve[%, Union@Cases[%, _TorsionPDNP, Infinity]] // Flatten**

Out[161]= $\{T^1{}_{12} \to \gamma + \overline{\gamma}, T^1{}_{13} \to \overline{\alpha} + \beta - \overline{\pi}, T^1{}_{14} \to \alpha + \overline{\beta} - \pi, T^1{}_{23} \to -\overline{\nu}, T^1{}_{24} \to -\nu, T^1{}_{34} \to -\mu + \overline{\mu},$
$T^2{}_{12} \to \epsilon + \overline{\epsilon}, T^2{}_{13} \to \kappa, T^2{}_{14} \to \overline{\kappa}, T^2{}_{23} \to -\overline{\alpha} - \beta + \tau, T^2{}_{24} \to -\alpha - \overline{\beta} + \overline{\tau}, T^2{}_{34} \to -\rho + \overline{\rho},$
$T^3{}_{12} \to -\pi - \overline{\tau}, T^3{}_{13} \to -\epsilon + \overline{\epsilon} - \overline{\rho}, T^3{}_{14} \to -\overline{\sigma}, T^3{}_{23} \to -\gamma + \overline{\gamma} + \mu, T^3{}_{24} \to \lambda, T^3{}_{34} \to \alpha - \overline{\beta},$
$T^4{}_{12} \to -\overline{\pi} - \tau, T^4{}_{13} \to -\sigma, T^4{}_{14} \to \epsilon - \overline{\epsilon} - \rho, T^4{}_{23} \to \overline{\lambda}, T^4{}_{24} \to \gamma - \overline{\gamma} + \overline{\mu}, T^4{}_{34} \to -\overline{\alpha} + \beta\}$

We store these values.

In[162]:= **% /. Rule → ComponentValue;**

In[163]:= **RuleToSet@TorsionPDNP;**

We can now write down the equations arising from the commutation relations. First we define a scalar function upon which the differential operators will act.

In[164]:= **DefTensor[ψ[], M4]**

** DefTensor: Defining tensor ψ[].

By default, `xTensor`` commutes automatically pairs of covariant derivatives when they act on scalars.

In[165]:= **PDNP[-a]@PDNP[-b]@ψ[]**

Out[165]= $\mathcal{D}_b \, \mathcal{D}_a \, \psi + T^c{}_{ba} \, (\mathcal{D}_c \, \psi)$

We need to remove this automatic commutation in our calculations. This is done by means of the next line of code.

In[166]:= $\mathcal{D}_{\underline{\text{xAct`xTensor`Private`b\$}}}$ $\mathcal{D}_{\underline{\text{xAct`xTensor`Private`a\$}}}$ `xAct`xTensor`Private`scalar$_Symbol?xTensorQ[] =.`

Commutation of the operator `PDNP` on the scalar ψ.

In[167]:= `PDNP[-a]@PDNP[-b]@ψ[] - PDNP[-b]@PDNP[-a]@ψ[] ==`
`SortCovDs[PDNP[-a]@PDNP[-b]@ψ[] - PDNP[-b]@PDNP[-a]@ψ[]]`

Out[167]= $\mathcal{D}_a\,\mathcal{D}_b\,\psi\ -\mathcal{D}_b\,\mathcal{D}_a\,\psi\ == T^c{}_{ba}\,(\mathcal{D}_c\,\psi\ )$

The components of this relation in the null tetrad yield the Newman-Penrose commutation relations.

In[168]:= `ToBasis[NP] /@%`

Out[168]= $e_b{}^c\,(\mathcal{D}_a\,\mathcal{D}_c\,\psi\ ) - e_a{}^c\,(\mathcal{D}_b\,\mathcal{D}_c\,\psi\ ) == T^c{}_{ba}\,(\mathcal{D}_c\,\psi\ )$

In[169]:= `ToBasis[NP] /@%`

Out[169]= $\mathcal{D}_a\,\mathcal{D}_b\,\psi\ -\mathcal{D}_b\,\mathcal{D}_a\,\psi\ == T^c{}_{ba}\,(\mathcal{D}_c\,\psi\ )$

In[170]:= `TraceBasisDummy /@%`

Out[170]= $\mathcal{D}_a\,\mathcal{D}_b\,\psi\ -\mathcal{D}_b\,\mathcal{D}_a\,\psi\ == T^1{}_{ba}\,(D\,\psi\ ) + T^2{}_{ba}\,(\Delta\,\psi\ ) + T^3{}_{ba}\,(\delta\,\psi\ ) + T^4{}_{ba}\,(\overline{\delta}\,\psi\ )$

In[171]:= `Union@Flatten@ComponentArray@%`

Out[171]= $\Big\{$True, $D\,\Delta\,\psi\ -\Delta\,D\,\psi\ == (-\gamma\ -\overline{\gamma}\ )\,(D\,\psi\ ) + (-\epsilon\ -\overline{\epsilon}\ )\,(\Delta\,\psi\ ) + (\pi\ +\overline{\tau}\ )\,(\delta\,\psi\ ) + (\overline{\pi}\ +\tau\ )\,(\overline{\delta}\,\psi\ )$,

$-(D\,\Delta\,\psi\ ) + \Delta\,D\,\psi\ == (\gamma\ +\overline{\gamma}\ )\,(D\,\psi\ ) + (\epsilon\ +\overline{\epsilon}\ )\,(\Delta\,\psi\ ) + (-\pi\ -\overline{\tau}\ )\,(\delta\,\psi\ ) + (-\overline{\pi}\ -\tau\ )\,(\overline{\delta}\,\psi\ )$,

$D\,\delta\,\psi\ -\delta\,D\,\psi\ == (-\overline{\alpha}\ -\beta\ +\overline{\pi}\ )\,(D\,\psi\ ) -\kappa\ \,(\Delta\,\psi\ ) + (\epsilon\ -\overline{\epsilon}\ +\overline{\rho}\ )\,(\delta\,\psi\ ) + \sigma\ \,(\overline{\delta}\,\psi\ )$,

$-(D\,\delta\,\psi\ ) + \delta\,D\,\psi\ == (\overline{\alpha}\ +\beta\ -\overline{\pi}\ )\,(D\,\psi\ ) + \kappa\ \,(\Delta\,\psi\ ) + (-\epsilon\ +\overline{\epsilon}\ -\overline{\rho}\ )\,(\delta\,\psi\ ) -\sigma\ \,(\overline{\delta}\,\psi\ )$,

$\Delta\,\delta\,\psi\ -\delta\,\Delta\,\psi\ == \overline{\nu}\ \,(D\,\psi\ ) + (\overline{\alpha}\ +\beta\ -\tau\ )\,(\Delta\,\psi\ ) + (\gamma\ -\overline{\gamma}\ -\mu\ )\,(\delta\,\psi\ ) -\overline{\lambda}\ \,(\overline{\delta}\,\psi\ )$,

$-(\Delta\,\delta\,\psi\ ) + \delta\,\Delta\,\psi\ == -\overline{\nu}\ \,(D\,\psi\ ) + (-\overline{\alpha}\ -\beta\ +\tau\ )\,(\Delta\,\psi\ ) + (-\gamma\ +\overline{\gamma}\ +\mu\ )\,(\delta\,\psi\ ) + \overline{\lambda}\ \,(\overline{\delta}\,\psi\ )$,

$D\,\overline{\delta}\,\psi\ -\overline{\delta}\,D\,\psi\ == \left(-\alpha\ -\overline{\beta}\ +\pi\ \right)\,(D\,\psi\ ) -\overline{\kappa}\ \,(\Delta\,\psi\ ) + \overline{\sigma}\ \,(\delta\,\psi\ ) + (-\epsilon\ +\overline{\epsilon}\ +\rho\ )\,(\overline{\delta}\,\psi\ )$,

$-\left(D\,\overline{\delta}\,\psi\ \right) + \overline{\delta}\,D\,\psi\ == \left(\alpha\ +\overline{\beta}\ -\pi\ \right)\,(D\,\psi\ ) + \overline{\kappa}\ \,(\Delta\,\psi\ ) -\overline{\sigma}\ \,(\delta\,\psi\ ) + (\epsilon\ -\overline{\epsilon}\ -\rho\ )\,(\overline{\delta}\,\psi\ )$,

$\Delta\,\overline{\delta}\,\psi\ -\overline{\delta}\,\Delta\,\psi\ == \nu\ \,(D\,\psi\ ) + \left(\alpha\ +\overline{\beta}\ -\overline{\tau}\ \right)\,(\Delta\,\psi\ ) -\lambda\ \,(\delta\,\psi\ ) + (-\gamma\ +\overline{\gamma}\ -\overline{\mu}\ )\,(\overline{\delta}\,\psi\ )$,

$-\left(\Delta\,\overline{\delta}\,\psi\ \right) + \overline{\delta}\,\Delta\,\psi\ == -\nu\ \,(D\,\psi\ ) + \left(-\alpha\ -\overline{\beta}\ +\overline{\tau}\ \right)\,(\Delta\,\psi\ ) + \lambda\ \,(\delta\,\psi\ ) + (\gamma\ -\overline{\gamma}\ +\overline{\mu}\ )\,(\overline{\delta}\,\psi\ )$,

$\delta\,\overline{\delta}\,\psi\ -\overline{\delta}\,\delta\,\psi\ == (\mu\ -\overline{\mu}\ )\,(D\,\psi\ ) + (\rho\ -\overline{\rho}\ )\,(\Delta\,\psi\ ) + \left(-\alpha\ +\overline{\beta}\ \right)\,(\delta\,\psi\ ) + (\overline{\alpha}\ -\beta\ )\,(\overline{\delta}\,\psi\ )$,

$-\left(\delta\,\overline{\delta}\,\psi\ \right) + \overline{\delta}\,\delta\,\psi\ == (-\mu\ +\overline{\mu}\ )\,(D\,\psi\ ) + (-\rho\ +\overline{\rho}\ )\,(\Delta\,\psi\ ) + \left(\alpha\ -\overline{\beta}\ \right)\,(\delta\,\psi\ ) + (-\overline{\alpha}\ +\beta\ )\,(\overline{\delta}\,\psi\ )\Big\}$

We extract the independent equations from this set.

In[172]:= `Solve[%, Union@Cases[%, PDNP[_]@PDNP[_]@ψ[], Infinity]] // Flatten`

Solve::svars : Equations may not give solutions for all "solve" variables. »

Out[172]= $\Big\{D\,\Delta\,\psi\ \to (-\gamma\ -\overline{\gamma}\ )\,(D\,\psi\ ) + (-\epsilon\ -\overline{\epsilon}\ )\,(\Delta\,\psi\ ) + \Delta\,D\,\psi\ + (\pi\ +\overline{\tau}\ )\,(\delta\,\psi\ ) + (\overline{\pi}\ +\tau\ )\,(\overline{\delta}\,\psi\ )$,

$D\,\delta\,\psi\ \to (-\overline{\alpha}\ -\beta\ +\overline{\pi}\ )\,(D\,\psi\ ) -\kappa\ \,(\Delta\,\psi\ ) + (\epsilon\ -\overline{\epsilon}\ +\overline{\rho}\ )\,(\delta\,\psi\ ) + \delta\,D\,\psi\ + \sigma\ \,(\overline{\delta}\,\psi\ )$,

$D\,\overline{\delta}\,\psi\ \to \left(-\alpha\ -\overline{\beta}\ +\pi\ \right)\,(D\,\psi\ ) -\overline{\kappa}\ \,(\Delta\,\psi\ ) + \overline{\sigma}\ \,(\delta\,\psi\ ) + (-\epsilon\ +\overline{\epsilon}\ +\rho\ )\,(\overline{\delta}\,\psi\ ) + \overline{\delta}\,D\,\psi\ $,

$\Delta\,\delta\,\psi\ \to \overline{\nu}\ \,(D\,\psi\ ) + (\overline{\alpha}\ +\beta\ -\tau\ )\,(\Delta\,\psi\ ) + (\gamma\ -\overline{\gamma}\ -\mu\ )\,(\delta\,\psi\ ) + \delta\,\Delta\,\psi\ -\overline{\lambda}\ \,(\overline{\delta}\,\psi\ )$,

$\Delta\,\overline{\delta}\,\psi\ \to \nu\ \,(D\,\psi\ ) + \left(\alpha\ +\overline{\beta}\ -\overline{\tau}\ \right)\,(\Delta\,\psi\ ) -\lambda\ \,(\delta\,\psi\ ) + (-\gamma\ +\overline{\gamma}\ -\overline{\mu}\ )\,(\overline{\delta}\,\psi\ ) + \overline{\delta}\,\Delta\,\psi\ $,

$\delta\,\overline{\delta}\,\psi\ \to (\mu\ -\overline{\mu}\ )\,(D\,\psi\ ) + (\rho\ -\overline{\rho}\ )\,(\Delta\,\psi\ ) + \left(-\alpha\ +\overline{\beta}\ \right)\,(\delta\,\psi\ ) + (\overline{\alpha}\ -\beta\ )\,(\overline{\delta}\,\psi\ ) + \overline{\delta}\,\delta\,\psi\ \Big\}$

These equations come in complex-conjugate pairs. Therefore we only need to consider a subset of them.

In[173]:= **Expand //@ %;**

In[174]:= **Union[%, SameTest → (Dagger /@ #1 === #2 &)];**

In[175]:= **FullSimplify //@ %**

Out[175]= $\left\{ D \triangle \psi \;\to\; -(\gamma + \overline{\gamma})\,(D\psi) - (\epsilon + \overline{\epsilon})\,(\triangle\psi) + \triangle D\psi + (\pi + \overline{\tau})\,(\delta\psi) + (\overline{\pi} + \tau)\,(\overline{\delta}\psi), \right.$

$D\,\delta\psi \;\to\; -(\overline{\alpha} + \beta - \overline{\pi})\,(D\psi) - \kappa\,(\triangle\psi) + (\epsilon - \overline{\epsilon} + \overline{\rho})\,(\delta\psi) + \delta D\psi + \sigma\,(\overline{\delta}\psi),$

$\triangle\,\delta\psi \;\to\; \overline{\nu}\,(D\psi) + (\overline{\alpha} + \beta - \tau)\,(\triangle\psi) + (\gamma - \overline{\gamma} - \mu)\,(\delta\psi) + \delta\triangle\psi - \overline{\lambda}\,(\overline{\delta}\psi),$

$\left. \delta\,\overline{\delta}\psi \;\to\; (\mu - \overline{\mu})\,(D\psi) + (\rho - \overline{\rho})\,(\triangle\psi) + \left(-\alpha + \overline{\beta}\right)\,(\delta\psi) + (\overline{\alpha} - \beta)\,\left(\overline{\delta}\psi\right) + \overline{\delta}\,\delta\psi \right\}$

Therefore, the full set of independent commutation relations is

In[176]:= **NPCommutators = % /. Rule -> Equal**

Out[176]= $\left\{ D \triangle \psi \;==\; -(\gamma + \overline{\gamma})\,(D\psi) - (\epsilon + \overline{\epsilon})\,(\triangle\psi) + \triangle D\psi + (\pi + \overline{\tau})\,(\delta\psi) + (\overline{\pi} + \tau)\,(\overline{\delta}\psi), \right.$

$D\,\delta\psi \;==\; -(\overline{\alpha} + \beta - \overline{\pi})\,(D\psi) - \kappa\,(\triangle\psi) + (\epsilon - \overline{\epsilon} + \overline{\rho})\,(\delta\psi) + \delta D\psi + \sigma\,(\overline{\delta}\psi),$

$\triangle\,\delta\psi \;==\; \overline{\nu}\,(D\psi) + (\overline{\alpha} + \beta - \tau)\,(\triangle\psi) + (\gamma - \overline{\gamma} - \mu)\,(\delta\psi) + \delta\triangle\psi - \overline{\lambda}\,(\overline{\delta}\psi),$

$\left. \delta\,\overline{\delta}\psi \;==\; (\mu - \overline{\mu})\,(D\psi) + (\rho - \overline{\rho})\,(\triangle\psi) + \left(-\alpha + \overline{\beta}\right)\,(\delta\psi) + (\overline{\alpha} - \beta)\,\left(\overline{\delta}\psi\right) + \overline{\delta}\,\delta\psi \right\}$

■ **Components of the Riemann tensor in terms of the components of the curvature spinors**

We are going to write the components of the Riemann tensor in the null tetrad in terms of the spinor curvature components expressed in the spin dyad. First we compute folded rules containing the independent components of each of the quantities involved.

In[177]:= **AllComponentValues@ToBasis[Dyad]@PsiCDe[-A, -B, -C, -D]**

Out[177]= FoldedRule[$\{\Psi[\nabla]_{0010} \to \Psi[\nabla]_{0001}, \Psi[\nabla]_{0100} \to \Psi[\nabla]_{0001}, \Psi[\nabla]_{0101} \to \Psi[\nabla]_{0011}, \Psi[\nabla]_{0110} \to \Psi[\nabla]_{0011},$
$\Psi[\nabla]_{1000} \to \Psi[\nabla]_{0001}, \Psi[\nabla]_{1001} \to \Psi[\nabla]_{0011}, \Psi[\nabla]_{1010} \to \Psi[\nabla]_{0011}, \Psi[\nabla]_{1011} \to \Psi[\nabla]_{0111},$
$\Psi[\nabla]_{1100} \to \Psi[\nabla]_{0011}, \Psi[\nabla]_{1101} \to \Psi[\nabla]_{0111}, \Psi[\nabla]_{1110} \to \Psi[\nabla]_{0111}\}, \{\Psi[\nabla]_{0000} \to \Psi[\nabla]_{0000},$
$\Psi[\nabla]_{0001} \to \Psi[\nabla]_{0001}, \Psi[\nabla]_{0011} \to \Psi[\nabla]_{0011}, \Psi[\nabla]_{0111} \to \Psi[\nabla]_{0111}, \Psi[\nabla]_{1111} \to \Psi[\nabla]_{1111}\}$]

In[178]:= **RuleToSet@PsiCDe**

In[179]:= **AllComponentValues@ToBasis[Dyad†]@PsiCDe†[-A†, -B†, -C†, -D†]**

Out[179]= FoldedRule[$\{\Psi[\nabla]\dagger_{0'0'1'0'} \to \Psi[\nabla]\dagger_{0'0'0'1'}, \Psi[\nabla]\dagger_{0'1'0'0'} \to \Psi[\nabla]\dagger_{0'0'0'1'},$
$\Psi[\nabla]\dagger_{0'1'0'1'} \to \Psi[\nabla]\dagger_{0'0'1'1'}, \Psi[\nabla]\dagger_{0'1'1'0'} \to \Psi[\nabla]\dagger_{0'0'1'1'}, \Psi[\nabla]\dagger_{1'0'0'0'} \to \Psi[\nabla]\dagger_{0'0'0'1'},$
$\Psi[\nabla]\dagger_{1'0'0'1'} \to \Psi[\nabla]\dagger_{0'0'1'1'}, \Psi[\nabla]\dagger_{1'0'1'0'} \to \Psi[\nabla]\dagger_{0'0'1'1'}, \Psi[\nabla]\dagger_{1'0'1'1'} \to \Psi[\nabla]\dagger_{0'1'1'1'},$
$\Psi[\nabla]\dagger_{1'1'0'0'} \to \Psi[\nabla]\dagger_{0'0'1'1'}, \Psi[\nabla]\dagger_{1'1'0'1'} \to \Psi[\nabla]\dagger_{0'1'1'1'}, \Psi[\nabla]\dagger_{1'1'1'0'} \to \Psi[\nabla]\dagger_{0'1'1'1'}\},$
$\{\Psi[\nabla]\dagger_{0'0'0'0'} \to \Psi[\nabla]\dagger_{0'0'0'0'}, \Psi[\nabla]\dagger_{0'0'0'1'} \to \Psi[\nabla]\dagger_{0'0'0'1'}, \Psi[\nabla]\dagger_{0'0'1'1'} \to \Psi[\nabla]\dagger_{0'0'1'1'},$
$\Psi[\nabla]\dagger_{0'1'1'1'} \to \Psi[\nabla]\dagger_{0'1'1'1'}, \Psi[\nabla]\dagger_{1'1'1'1'} \to \Psi[\nabla]\dagger_{1'1'1'1'}\}$]

In[180]:= **RuleToSet@PsiCDe†**

In[181]:= **AllComponentValues@ToBasis[Dyad†]@ToBasis[Dyad]@PhiCDe[-A, -B, -A†, -B†]**

Out[181]= FoldedRule[$\{\Phi[\nabla]_{001'0'} \to \Phi[\nabla]_{000'1'},$
$\Phi[\nabla]_{011'0'} \to \Phi[\nabla]_{010'1'}, \Phi[\nabla]_{100'0'} \to \Phi[\nabla]_{010'0'}, \Phi[\nabla]_{100'1'} \to \Phi[\nabla]_{010'1'},$
$\Phi[\nabla]_{101'0'} \to \Phi[\nabla]_{010'1'}, \Phi[\nabla]_{101'1'} \to \Phi[\nabla]_{011'1'}, \Phi[\nabla]_{111'0'} \to \Phi[\nabla]_{110'1'}\},$
$\{\Phi[\nabla]_{000'0'} \to \Phi[\nabla]_{000'0'}, \Phi[\nabla]_{000'1'} \to \Phi[\nabla]_{000'1'}, \Phi[\nabla]_{001'1'} \to \Phi[\nabla]_{001'1'},$
$\Phi[\nabla]_{010'0'} \to \Phi[\nabla]_{010'0'}, \Phi[\nabla]_{010'1'} \to \Phi[\nabla]_{010'1'}, \Phi[\nabla]_{011'1'} \to \Phi[\nabla]_{011'1'},$
$\Phi[\nabla]_{110'0'} \to \Phi[\nabla]_{110'0'}, \Phi[\nabla]_{110'1'} \to \Phi[\nabla]_{110'1'}, \Phi[\nabla]_{111'1'} \to \Phi[\nabla]_{111'1'}\}$]

In[182]:= **RuleToSet@PhiCDe**

We introduce the scalars which will represent the components of the Weyl spinor

```
In[183]:= DefTensor[#, M4, Dagger → Complex] & /@
          {PsiCDe0Dyad[], PsiCDe1Dyad[], PsiCDe2Dyad[], PsiCDe3Dyad[], PsiCDe4Dyad[]};
```

** DefTensor: Defining tensor PsiCDe0Dyad[].

** DefTensor: Defining tensor PsiCDe0Dyad†[].

** DefTensor: Defining tensor PsiCDe1Dyad[].

** DefTensor: Defining tensor PsiCDe1Dyad†[].

** DefTensor: Defining tensor PsiCDe2Dyad[].

** DefTensor: Defining tensor PsiCDe2Dyad†[].

** DefTensor: Defining tensor PsiCDe3Dyad[].

** DefTensor: Defining tensor PsiCDe3Dyad†[].

** DefTensor: Defining tensor PsiCDe4Dyad[].

** DefTensor: Defining tensor PsiCDe4Dyad†[].

The different components of the Weyl spinor are formatted with their conventional names.

```
In[184]:= PrintAs@PsiCDe0Dyad ^= "Ψ₀";
          PrintAs@PsiCDe1Dyad ^= "Ψ₁";
          PrintAs@PsiCDe2Dyad ^= "Ψ₂";
          PrintAs@PsiCDe3Dyad ^= "Ψ₃";
          PrintAs@PsiCDe4Dyad ^= "Ψ₄";
          PrintAs@PsiCDe0Dyad† ^= "Ψ̄₀";
          PrintAs@PsiCDe1Dyad† ^= "Ψ̄₁";
          PrintAs@PsiCDe2Dyad† ^= "Ψ̄₂";
          PrintAs@PsiCDe3Dyad† ^= "Ψ̄₃";
          PrintAs@PsiCDe4Dyad† ^= "Ψ̄₄";
```

```
In[194]:= SetToRule@PsiCDe
```

```
In[195]:= SetToRule@PsiCDe†
```

Explicit definition of the components of the Weyl spinor.

```
In[196]:= {PsiCDe[{0, -Dyad}, {0, -Dyad}, {0, -Dyad}, {0, -Dyad}] -> PsiCDe0Dyad[],
           PsiCDe[{0, -Dyad}, {0, -Dyad}, {0, -Dyad}, {1, -Dyad}] -> PsiCDe1Dyad[],
           PsiCDe[{0, -Dyad}, {0, -Dyad}, {1, -Dyad}, {1, -Dyad}] -> PsiCDe2Dyad[],
           PsiCDe[{0, -Dyad}, {1, -Dyad}, {1, -Dyad}, {1, -Dyad}] -> PsiCDe3Dyad[],
           PsiCDe[{1, -Dyad}, {1, -Dyad}, {1, -Dyad}, {1, -Dyad}] -> PsiCDe4Dyad[]}
```

$$Out[196]= \{ \Psi[\nabla]_{0000} \to \Psi_0 , \Psi[\nabla]_{0001} \to \Psi_1 , \Psi[\nabla]_{0011} \to \Psi_2 , \Psi[\nabla]_{0111} \to \Psi_3 , \Psi[\nabla]_{1111} \to \Psi_4 \}$$

```
In[197]:= % /. Rule → ComponentValue
```

$$Out[197]= \{ \Psi[\nabla]_{0000} \to \Psi_0 , \Psi[\nabla]_{0001} \to \Psi_1 , \Psi[\nabla]_{0011} \to \Psi_2 , \Psi[\nabla]_{0111} \to \Psi_3 , \Psi[\nabla]_{1111} \to \Psi_4 \}$$

```
In[198]:= Map[Dagger, %, {2}]
```

$$Out[198]= \{ \Psi[\nabla]\dagger_{0'0'0'0'} \to \overline{\Psi}_0 , \Psi[\nabla]\dagger_{0'0'0'1'} \to \overline{\Psi}_1 ,$$
$$\Psi[\nabla]\dagger_{0'0'1'1'} \to \overline{\Psi}_2 , \Psi[\nabla]\dagger_{0'1'1'1'} \to \overline{\Psi}_3 , \Psi[\nabla]\dagger_{1'1'1'1'} \to \overline{\Psi}_4 \}$$

In[199]:= `% /. Rule → ComponentValue`

Out[199]= $\left\{ \Psi[\nabla]\dagger_{0'\,0'\,0'\,0'} \to \overline{\Psi}_0 \ , \ \Psi[\nabla]\dagger_{0'\,0'\,0'\,1'} \to \overline{\Psi}_1 \ , \right.$
$\left. \Psi[\nabla]\dagger_{0'\,0'\,1'\,1'} \to \overline{\Psi}_2 \ , \ \Psi[\nabla]\dagger_{0'\,1'\,1'\,1'} \to \overline{\Psi}_3 \ , \ \Psi[\nabla]\dagger_{1'\,1'\,1'\,1'} \to \overline{\Psi}_4 \ \right\}$

In[200]:= `RuleToSet@PsiCDe`

In[201]:= `RuleToSet@PsiCDe†`

We introduce the scalars which will represent the components of the Ricci spinor.

In[202]:=
```
DefTensor[♯, M4, Dagger → Complex] & /@
   {PhiCDe00Dyad[], PhiCDe01Dyad[], PhiCDe02Dyad[], PhiCDe10Dyad[],
    PhiCDe11Dyad[], PhiCDe12Dyad[], PhiCDe20Dyad[], PhiCDe21Dyad[], PhiCDe22Dyad[]};
```

** DefTensor: Defining tensor PhiCDe00Dyad[].

** DefTensor: Defining tensor PhiCDe00Dyad†[].

** DefTensor: Defining tensor PhiCDe01Dyad[].

** DefTensor: Defining tensor PhiCDe01Dyad†[].

** DefTensor: Defining tensor PhiCDe02Dyad[].

** DefTensor: Defining tensor PhiCDe02Dyad†[].

** DefTensor: Defining tensor PhiCDe10Dyad[].

** DefTensor: Defining tensor PhiCDe10Dyad†[].

** DefTensor: Defining tensor PhiCDe11Dyad[].

** DefTensor: Defining tensor PhiCDe11Dyad†[].

** DefTensor: Defining tensor PhiCDe12Dyad[].

** DefTensor: Defining tensor PhiCDe12Dyad†[].

** DefTensor: Defining tensor PhiCDe20Dyad[].

** DefTensor: Defining tensor PhiCDe20Dyad†[].

** DefTensor: Defining tensor PhiCDe21Dyad[].

** DefTensor: Defining tensor PhiCDe21Dyad†[].

** DefTensor: Defining tensor PhiCDe22Dyad[].

** DefTensor: Defining tensor PhiCDe22Dyad†[].

The components of the Ricci spinor are formatted with their conventional names.

In[203]:=
```
PrintAs[PhiCDe00Dyad] ^= "Φ₀₀";
PrintAs[PhiCDe01Dyad] ^= "Φ₀₁";
PrintAs[PhiCDe02Dyad] ^= "Φ₀₂";
PrintAs[PhiCDe10Dyad] ^= "Φ₁₀";
PrintAs[PhiCDe11Dyad] ^= "Φ₁₁";
PrintAs[PhiCDe12Dyad] ^= "Φ₁₂";
PrintAs[PhiCDe20Dyad] ^= "Φ₂₀";
PrintAs[PhiCDe21Dyad] ^= "Φ₂₁";
PrintAs[PhiCDe22Dyad] ^= "Φ₂₂";
```

In[212]:= `SetToRule@PhiCDe`

Explicit definition of the components of the Ricci spinor.

In[213]:= `{PhiCDe[{0, -Dyad}, {0, -Dyad}, {0, -Dyad†}, {0, -Dyad†}] → PhiCDe00Dyad[],`
`PhiCDe[{0, -Dyad}, {0, -Dyad}, {0, -Dyad†}, {1, -Dyad†}] → PhiCDe01Dyad[],`
`PhiCDe[{0, -Dyad}, {0, -Dyad}, {1, -Dyad†}, {1, -Dyad†}] → PhiCDe02Dyad[],`
`PhiCDe[{0, -Dyad}, {1, -Dyad}, {0, -Dyad†}, {0, -Dyad†}] → PhiCDe10Dyad[],`
`PhiCDe[{1, -Dyad}, {1, -Dyad}, {0, -Dyad†}, {0, -Dyad†}] → PhiCDe20Dyad[],`
`PhiCDe[{0, -Dyad}, {1, -Dyad}, {0, -Dyad†}, {1, -Dyad†}] → PhiCDe11Dyad[],`
`PhiCDe[{1, -Dyad}, {1, -Dyad}, {0, -Dyad†}, {1, -Dyad†}] → PhiCDe21Dyad[],`
`PhiCDe[{0, -Dyad}, {1, -Dyad}, {1, -Dyad†}, {1, -Dyad†}] → PhiCDe12Dyad[],`
`PhiCDe[{1, -Dyad}, {1, -Dyad}, {1, -Dyad†}, {1, -Dyad†}] → PhiCDe22Dyad[]}`

Out[213]= $\{ \Phi[\nabla]_{0\,0\,0'0'} \to \Phi_{00}$, $\Phi[\nabla]_{0\,0\,0'1'} \to \Phi_{01}$, $\Phi[\nabla]_{0\,0\,1'1'} \to \Phi_{02}$, $\Phi[\nabla]_{0\,1\,0'0'} \to \Phi_{10}$,
$\Phi[\nabla]_{1\,1\,0'0'} \to \Phi_{20}$, $\Phi[\nabla]_{0\,1\,0'1'} \to \Phi_{11}$, $\Phi[\nabla]_{1\,1\,0'1'} \to \Phi_{21}$, $\Phi[\nabla]_{0\,1\,1'1'} \to \Phi_{12}$, $\Phi[\nabla]_{1\,1\,1'1'} \to \Phi_{22}\}$

In[214]:= `% /. Rule → ComponentValue;`

In[215]:= `ToValues@Map[Dagger, %, {2}]`

Out[215]= $\{ \Phi_{00} \to$ `PhiCDe00Dyad†` , $\Phi_{10} \to$ `PhiCDe01Dyad†` , $\Phi_{20} \to$ `PhiCDe02Dyad†` ,
$\Phi_{01} \to$ `PhiCDe10Dyad†` , $\Phi_{02} \to$ `PhiCDe20Dyad†` , $\Phi_{11} \to$ `PhiCDe11Dyad†` ,
$\Phi_{12} \to$ `PhiCDe21Dyad†` , $\Phi_{21} \to$ `PhiCDe12Dyad†` , $\Phi_{22} \to$ `PhiCDe22Dyad†` $\}$

In[216]:= `% /. Rule → Equal`

Out[216]= $\{ \Phi_{00} ==$ `PhiCDe00Dyad†` , $\Phi_{10} ==$ `PhiCDe01Dyad†` , $\Phi_{20} ==$ `PhiCDe02Dyad†` ,
$\Phi_{01} ==$ `PhiCDe10Dyad†` , $\Phi_{02} ==$ `PhiCDe20Dyad†` , $\Phi_{11} ==$ `PhiCDe11Dyad†` ,
$\Phi_{12} ==$ `PhiCDe21Dyad†` , $\Phi_{21} ==$ `PhiCDe12Dyad†` , $\Phi_{22} ==$ `PhiCDe22Dyad†` $\}$

In[217]:= `Solve[%,`
`{PhiCDe00Dyad†[], PhiCDe01Dyad†[], PhiCDe02Dyad†[], PhiCDe10Dyad†[], PhiCDe11Dyad†[],`
`PhiCDe12Dyad†[], PhiCDe20Dyad†[], PhiCDe21Dyad†[], PhiCDe22Dyad†[]}] // Flatten`

Out[217]= $\{$ `PhiCDe00Dyad†` $\to \Phi_{00}$, `PhiCDe01Dyad†` $\to \Phi_{10}$, `PhiCDe02Dyad†` $\to \Phi_{20}$,
`PhiCDe10Dyad†` $\to \Phi_{01}$, `PhiCDe20Dyad†` $\to \Phi_{02}$, `PhiCDe11Dyad†` $\to \Phi_{11}$,
`PhiCDe21Dyad†` $\to \Phi_{12}$, `PhiCDe12Dyad†` $\to \Phi_{21}$, `PhiCDe22Dyad†` $\to \Phi_{22}$ $\}$

In[218]:= `% /. Rule → Set;`

In[219]:= `RuleToSet@PhiCDe`

In[220]:= `PrintAs[LambdaCDe] ^= "Λ";`

Computation of the folded rule for the Riemann tensor.

In[221]:= `AllComponentValues@ToBasis[NP]@RiemannCD[-a, -b, -c, -d];`

In[222]:= `RuleToSet@RiemannCD`

Next we show how to write Riemann tensor components in terms of the components of curvature spinors

In[223]:= `RiemannCD[-a, -b, -c, -d]`

Out[223]= $R[\nabla]_{abcd}$

In[224]:= `% == Expand@Decomposition[SeparateSolderingForm[][%], Riemann]`

Out[224]= $R[\nabla]_{abcd} == \Phi[\nabla] \dagger_{A'\,B'\,D'\,C'} \, \epsilon_{AB}\, \epsilon_{DC}\, \sigma_a{}^{AA'}\, \sigma_b{}^{BB'}\, \sigma_c{}^{CC'}\, \sigma_d{}^{DD'} - \Phi[\nabla]_{ABD'\,C'}\, \epsilon_{DC}\, \overline{\epsilon}_{A'\,B'}\, \sigma_a{}^{AA'}\, \sigma_b{}^{BB'}\, \sigma_c{}^{CC'}\, \sigma_d{}^{DD'} -$
$2\, \Lambda \; \epsilon_{AD}\, \epsilon_{BC}\, \overline{\epsilon}_{A'\,D'}\, \overline{\epsilon}_{B'\,C'}\, \sigma_a{}^{AA'}\, \sigma_b{}^{BB'}\, \sigma_c{}^{CC'}\, \sigma_d{}^{DD'} + 2\, \Lambda \; \epsilon_{AC}\, \epsilon_{BD}\, \overline{\epsilon}_{A'\,C'}\, \overline{\epsilon}_{B'\,D'}\, \sigma_a{}^{AA'}\, \sigma_b{}^{BB'}\, \sigma_c{}^{CC'}\, \sigma_d{}^{DD'} -$
$\Phi[\nabla]_{DCA'\,B'}\, \epsilon_{AB}\, \overline{\epsilon}_{D'\,C'}\, \sigma_a{}^{AA'}\, \sigma_b{}^{BB'}\, \sigma_c{}^{CC'}\, \sigma_d{}^{DD'} + \Phi[\nabla]_{ABDC}\, \overline{\epsilon}_{A'\,B'}\, \overline{\epsilon}_{D'\,C'}\, \sigma_a{}^{AA'}\, \sigma_b{}^{BB'}\, \sigma_c{}^{CC'}\, \sigma_d{}^{DD'}$

In[225]:= `ToBasis[Dyad†] /@ ToBasis[Dyad] /@ ToBasis[NP] /@ %`

Out[225]= $R[\nabla]_{abcd} == \Psi[\nabla]\,\dagger_{A'B'D'C'}\,\epsilon_{AB}\,\epsilon_{DC}\,\sigma_a{}^{AA'}\,\sigma_b{}^{BB'}\,\sigma_c{}^{CC'}\,\sigma_d{}^{DD'} - \Phi[\nabla]_{ABD'C'}\,\epsilon_{DC}\,\overline{\epsilon}_{A'B'}\,\sigma_a{}^{AA'}\,\sigma_b{}^{BB'}\,\sigma_c{}^{CC'}\,\sigma_d{}^{DD'} -$
$2\,\Lambda\ \epsilon_{AD}\,\epsilon_{BC}\,\overline{\epsilon}_{A'D'}\,\overline{\epsilon}_{B'C'}\,\sigma_a{}^{AA'}\,\sigma_b{}^{BB'}\,\sigma_c{}^{CC'}\,\sigma_d{}^{DD'} + 2\,\Lambda\ \epsilon_{AC}\,\epsilon_{BD}\,\overline{\epsilon}_{A'C'}\,\overline{\epsilon}_{B'D'}\,\sigma_a{}^{AA'}\,\sigma_b{}^{BB'}\,\sigma_c{}^{CC'}\,\sigma_d{}^{DD'} -$
$\Phi[\nabla]_{DCA'B'}\,\epsilon_{AB}\,\overline{\epsilon}_{D'C'}\,\sigma_a{}^{AA'}\,\sigma_b{}^{BB'}\,\sigma_c{}^{CC'}\,\sigma_d{}^{DD'} + \Psi[\nabla]_{ABDC}\,\overline{\epsilon}_{A'B'}\,\overline{\epsilon}_{D'C'}\,\sigma_a{}^{AA'}\,\sigma_b{}^{BB'}\,\sigma_c{}^{CC'}\,\sigma_d{}^{DD'}$

In[226]:= `TraceBasisDummy /@ %;`

In[227]:= `Flatten@ComponentArray@%;`

In[228]:= `Union@%`

Out[228]= $\{$True, $-R[\nabla]_{1212} == 2\,\Lambda\ - 2\,\Phi_{11}\ + \Psi_2\ + \overline{\Psi}_2$, $R[\nabla]_{1212} == -2\,\Lambda\ + 2\,\Phi_{11}\ - \Psi_2\ - \overline{\Psi}_2$,
$-R[\nabla]_{1213} == -\Phi_{01}\ + \Psi_1$, $R[\nabla]_{1213} == \Phi_{01}\ - \Psi_1$, $-R[\nabla]_{1214} == -\Phi_{10}\ + \overline{\Psi}_1$, $R[\nabla]_{1214} == \Phi_{10}\ - \overline{\Psi}_1$,
$-R[\nabla]_{1223} == \Phi_{12}\ - \overline{\Psi}_3$, $R[\nabla]_{1223} == -\Phi_{12}\ + \overline{\Psi}_3$, $-R[\nabla]_{1224} == \Phi_{21}\ - \Psi_3$, $R[\nabla]_{1224} == -\Phi_{21}\ + \Psi_3$,
$-R[\nabla]_{1234} == -\Psi_2\ + \overline{\Psi}_2$, $R[\nabla]_{1234} == \Psi_2\ - \overline{\Psi}_2$, $-R[\nabla]_{1313} == \Psi_0$, $R[\nabla]_{1313} == -\Psi_0$,
$-R[\nabla]_{1314} == -\Phi_{00}$, $R[\nabla]_{1314} == \Phi_{00}$, $-R[\nabla]_{1323} == \Phi_{02}$, $R[\nabla]_{1323} == -\Phi_{02}$, $-R[\nabla]_{1324} == 2\,\Lambda\ - \Psi_2$,
$R[\nabla]_{1324} == -2\,\Lambda\ + \Psi_2$, $-R[\nabla]_{1334} == -\Phi_{01}\ - \Psi_1$, $R[\nabla]_{1334} == \Phi_{01}\ + \Psi_1$, $-R[\nabla]_{1414} == \overline{\Psi}_0$,
$R[\nabla]_{1414} == -\overline{\Psi}_0$, $-R[\nabla]_{1423} == 2\,\Lambda\ - \overline{\Psi}_2$, $R[\nabla]_{1423} == -2\,\Lambda\ + \overline{\Psi}_2$, $-R[\nabla]_{1424} == \Phi_{20}$,
$R[\nabla]_{1424} == -\Phi_{20}$, $-R[\nabla]_{1434} == \Phi_{10}\ + \overline{\Psi}_1$, $R[\nabla]_{1434} == -\Phi_{10}\ - \overline{\Psi}_1$, $-R[\nabla]_{2323} == \overline{\Psi}_4$,
$R[\nabla]_{2323} == -\overline{\Psi}_4$, $-R[\nabla]_{2324} == -\Phi_{22}$, $R[\nabla]_{2324} == \Phi_{22}$, $-R[\nabla]_{2334} == -\Phi_{12}\ - \overline{\Psi}_3$,
$R[\nabla]_{2334} == \Phi_{12}\ + \overline{\Psi}_3$, $-R[\nabla]_{2424} == \Psi_4$, $R[\nabla]_{2424} == -\Psi_4$, $-R[\nabla]_{2434} == \Phi_{21}\ + \Psi_3$,
$R[\nabla]_{2434} == -\Phi_{21}\ - \Psi_3$, $-R[\nabla]_{3434} == 2\,\Lambda\ + 2\,\Phi_{11}\ + \Psi_2\ + \overline{\Psi}_2$, $R[\nabla]_{3434} == -2\,\Lambda\ - 2\,\Phi_{11}\ - \Psi_2\ - \overline{\Psi}_2$ $\}$

In[229]:= `Solve[%, Union@Cases[%, _RiemannCD, Infinity]] // Flatten`

Out[229]= $\{R[\nabla]_{1212} \to -2\,\Lambda\ + 2\,\Phi_{11}\ - \Psi_2\ - \overline{\Psi}_2$, $R[\nabla]_{1213} \to \Phi_{01}\ - \Psi_1$, $R[\nabla]_{1214} \to \Phi_{10}\ - \overline{\Psi}_1$,
$R[\nabla]_{1223} \to -\Phi_{12}\ + \overline{\Psi}_3$, $R[\nabla]_{1224} \to -\Phi_{21}\ + \Psi_3$, $R[\nabla]_{1234} \to \Psi_2\ - \overline{\Psi}_2$, $R[\nabla]_{1313} \to -\Psi_0$,
$R[\nabla]_{1314} \to \Phi_{00}$, $R[\nabla]_{1323} \to -\Phi_{02}$, $R[\nabla]_{1324} \to -2\,\Lambda\ + \Psi_2$, $R[\nabla]_{1334} \to \Phi_{01}\ + \Psi_1$, $R[\nabla]_{1414} \to -\overline{\Psi}_0$,
$R[\nabla]_{1423} \to -2\,\Lambda\ + \overline{\Psi}_2$, $R[\nabla]_{1424} \to -\Phi_{20}$, $R[\nabla]_{1434} \to -\Phi_{10}\ - \overline{\Psi}_1$, $R[\nabla]_{2323} \to -\overline{\Psi}_4$, $R[\nabla]_{2324} \to \Phi_{22}$,
$R[\nabla]_{2334} \to \Phi_{12}\ + \overline{\Psi}_3$, $R[\nabla]_{2424} \to -\Psi_4$, $R[\nabla]_{2434} \to -\Phi_{21}\ - \Psi_3$, $R[\nabla]_{3434} \to -2\,\Lambda\ - 2\,\Phi_{11}\ - \Psi_2\ - \overline{\Psi}_2$ $\}$

We add the values just found to the folded rule containg the components of the Riemann tensor.

In[230]:= `SetToRule@RiemannCD`

In[231]:= `%% /. Rule -> ComponentValue;`

In[232]:= `RuleToSet@RiemannCD`

From previous computations we can obtain and store the values for the components of the Ricci tensor.

In[233]:= `AllComponentValues@ToBasis[NP]@RicciCD[-a, -b];`

In[234]:= `RicciCD[-a, -b] == g[p, q] RiemannCD[-a, -p, -b, -q]`

Out[234]= $R[\nabla]_{ab} == g^{pq}\,R[\nabla]_{apbq}$

In[235]:= `ToBasis[NP] /@ %`

Out[235]= $R[\nabla]_{ab} == g^{cd}\,R[\nabla]_{acbd}$

In[236]:= `TraceBasisDummy /@ %`

Out[236]= $R[\nabla]_{ab} == R[\nabla]_{a1b2} + R[\nabla]_{a2b1} - R[\nabla]_{a3b4} - R[\nabla]_{a4b3}$

In[237]:= **Flatten@ComponentArray@%**

Out[237]= $\{R[\nabla]_{11} == -2\,\Phi_{00}$, $R[\nabla]_{12} == 6\,\Lambda\,-2\,\Phi_{11}$, $R[\nabla]_{13} == -2\,\Phi_{01}$, $R[\nabla]_{14} == -2\,\Phi_{10}$,
   $R[\nabla]_{21} == 6\,\Lambda\,-2\,\Phi_{11}$, $R[\nabla]_{22} == -2\,\Phi_{22}$, $R[\nabla]_{23} == -2\,\Phi_{12}$, $R[\nabla]_{24} == -2\,\Phi_{21}$,
   $R[\nabla]_{31} == -2\,\Phi_{01}$, $R[\nabla]_{32} == -2\,\Phi_{12}$, $R[\nabla]_{33} == -2\,\Phi_{02}$, $R[\nabla]_{34} == -6\,\Lambda\,-2\,\Phi_{11}$,
   $R[\nabla]_{41} == -2\,\Phi_{10}$, $R[\nabla]_{42} == -2\,\Phi_{21}$, $R[\nabla]_{43} == -6\,\Lambda\,-2\,\Phi_{11}$, $R[\nabla]_{44} == -2\,\Phi_{20}\}$

In[238]:= **Union@%**

Out[238]= $\{R[\nabla]_{11} == -2\,\Phi_{00}$, $R[\nabla]_{12} == 6\,\Lambda\,-2\,\Phi_{11}$, $R[\nabla]_{13} == -2\,\Phi_{01}$, $R[\nabla]_{14} == -2\,\Phi_{10}$,
   $R[\nabla]_{21} == 6\,\Lambda\,-2\,\Phi_{11}$, $R[\nabla]_{22} == -2\,\Phi_{22}$, $R[\nabla]_{23} == -2\,\Phi_{12}$, $R[\nabla]_{24} == -2\,\Phi_{21}$,
   $R[\nabla]_{31} == -2\,\Phi_{01}$, $R[\nabla]_{32} == -2\,\Phi_{12}$, $R[\nabla]_{33} == -2\,\Phi_{02}$, $R[\nabla]_{34} == -6\,\Lambda\,-2\,\Phi_{11}$,
   $R[\nabla]_{41} == -2\,\Phi_{10}$, $R[\nabla]_{42} == -2\,\Phi_{21}$, $R[\nabla]_{43} == -6\,\Lambda\,-2\,\Phi_{11}$, $R[\nabla]_{44} == -2\,\Phi_{20}\}$

In[239]:= **% /. Equal → ComponentValue;**

In[240]:= **RuleToSet@RicciCD**

The scalar curvature is worked out in a similar fashion.

In[241]:= **RicciScalarCD[] == ToBasis[NP]@g[a, b] RicciCD[-a, -b]**

Out[241]= $R[\nabla]\ == g^{ab}\,R[\nabla]_{ab}$

In[242]:= **ToBasis[NP] /@%**

Out[242]= $R[\nabla]\ == g^{ab}\,R[\nabla]_{ab}$

In[243]:= **TraceBasisDummy /@%**

Out[243]= $R[\nabla]\ == 24\,\Lambda$

In[244]:= **% /. Equal → ComponentValue**

Out[244]= $R[\nabla]\ \to 24\,\Lambda$

In[245]:= **RuleToSet@RicciScalarCD**

We may also need the components of the traceless part of the Ricci tensor.

In[246]:= **AllComponentValues@ToBasis[NP]@TFRicciCD[-a, -b];**

In[247]:= **TFRicciCD[-a, -b] == (TFRicciCD[-a, -b] // TFRicciToRicci)**

Out[247]= $S[\nabla]_{ab} == -6\,g_{ba}\,\Lambda\,+R[\nabla]_{ab}$

In[248]:= **ToBasis[NP] /@%**

Out[248]= $S[\nabla]_{ab} == -6\,g_{ba}\,\Lambda\,+R[\nabla]_{ab}$

In[249]:= **ToValues@Flatten@ComponentArray@%**

Out[249]= $\{S[\nabla]_{11} == -2\,\Phi_{00}$, $S[\nabla]_{12} == -2\,\Phi_{11}$, $S[\nabla]_{13} == -2\,\Phi_{01}$, $S[\nabla]_{14} == -2\,\Phi_{10}$,
   $S[\nabla]_{12} == -2\,\Phi_{11}$, $S[\nabla]_{22} == -2\,\Phi_{22}$, $S[\nabla]_{23} == -2\,\Phi_{12}$, $S[\nabla]_{24} == -2\,\Phi_{21}$,
   $S[\nabla]_{13} == -2\,\Phi_{01}$, $S[\nabla]_{23} == -2\,\Phi_{12}$, $S[\nabla]_{33} == -2\,\Phi_{02}$, $S[\nabla]_{34} == -2\,\Phi_{11}$,
   $S[\nabla]_{14} == -2\,\Phi_{10}$, $S[\nabla]_{24} == -2\,\Phi_{21}$, $S[\nabla]_{34} == -2\,\Phi_{11}$, $S[\nabla]_{44} == -2\,\Phi_{20}\}$

In[250]:= **Union@%**

Out[250]= $\{S[\nabla]_{11} == -2\,\Phi_{00}$, $S[\nabla]_{12} == -2\,\Phi_{11}$, $S[\nabla]_{13} == -2\,\Phi_{01}$, $S[\nabla]_{14} == -2\,\Phi_{10}$, $S[\nabla]_{22} == -2\,\Phi_{22}$,
   $S[\nabla]_{23} == -2\,\Phi_{12}$, $S[\nabla]_{24} == -2\,\Phi_{21}$, $S[\nabla]_{33} == -2\,\Phi_{02}$, $S[\nabla]_{34} == -2\,\Phi_{11}$, $S[\nabla]_{44} == -2\,\Phi_{20}\}$

In[251]:= `% /. Equal -> ComponentValue`

Out[251]= $\{S[\nabla]_{11} \to -2\,\Phi_{00}$, $S[\nabla]_{12} \to -2\,\Phi_{11}$, $S[\nabla]_{13} \to -2\,\Phi_{01}$, $S[\nabla]_{14} \to -2\,\Phi_{10}$, $S[\nabla]_{22} \to -2\,\Phi_{22}$, $S[\nabla]_{23} \to -2\,\Phi_{12}$, $S[\nabla]_{24} \to -2\,\Phi_{21}$, $S[\nabla]_{33} \to -2\,\Phi_{02}$, $S[\nabla]_{34} \to -2\,\Phi_{11}$, $S[\nabla]_{44} \to -2\,\Phi_{20}\}$

In[252]:= `RuleToSet@TFRicciCD`

This is the complete list of the components of the Riemann tensor.

In[253]:= `NPCVComponents = {PsiCDe0Dyad[], PsiCDe1Dyad[], PsiCDe2Dyad[], PsiCDe3Dyad[], PsiCDe4Dyad[],`
`    PhiCDe00Dyad[], PhiCDe01Dyad[], PhiCDe02Dyad[], PhiCDe10Dyad[], PhiCDe11Dyad[],`
`    PhiCDe12Dyad[], PhiCDe20Dyad[], PhiCDe21Dyad[], PhiCDe22Dyad[], LambdaCDe[]}`

Out[253]= $\{\Psi_0$, Ψ_1 , Ψ_2 , Ψ_3 , Ψ_4 , Φ_{00} , Φ_{01} , Φ_{02} , Φ_{10} , Φ_{11} , Φ_{12} , Φ_{20} , Φ_{21} , Φ_{22} , $\Lambda\}$

■ **The Newman-Penrose equations**

We now calculate the Riemann tensor with respect to the null tetrad

In[254]:= `RiemannCD[-a, -b, -c, -d] == ChangeCurvature[RiemannCD[-a, -b, -c, -d], CD, PDNP]`

`** DefTensor: Defining tensor ChristoffelCDPDNP[a, -b, -c].`

Out[254]= $R[\nabla]_{abcd} == g_{df}\left(\Gamma[\nabla,\mathcal{D}]^f{}_{bh}\,\Gamma[\nabla,\mathcal{D}]^h{}_{ac} - \Gamma[\nabla,\mathcal{D}]^f{}_{ah}\,\Gamma[\nabla,\mathcal{D}]^h{}_{bc} - \Gamma[\nabla,\mathcal{D}]^f{}_{hc}\,T^h{}_{ab} - \mathcal{D}_a\,\Gamma[\nabla,\mathcal{D}]^f{}_{bc} + \mathcal{D}_b\,\Gamma[\nabla,\mathcal{D}]^f{}_{ac}\right)$

In[255]:= `% /. ChristoffelCDPDNP -> ChristoffelCDePDNP`

Out[255]= $R[\nabla]_{abcd} == g_{df}\left(\Gamma[\nabla,\mathcal{D}]^f{}_{bh}\,\Gamma[\nabla,\mathcal{D}]^h{}_{ac} - \Gamma[\nabla,\mathcal{D}]^f{}_{ah}\,\Gamma[\nabla,\mathcal{D}]^h{}_{bc} - \Gamma[\nabla,\mathcal{D}]^f{}_{hc}\,T^h{}_{ab} - \mathcal{D}_a\,\Gamma[\nabla,\mathcal{D}]^f{}_{bc} + \mathcal{D}_b\,\Gamma[\nabla,\mathcal{D}]^f{}_{ac}\right)$

In[256]:= `ToBasis[NP] /@%`

Out[256]= $R[\nabla]_{abcd} == \Gamma[\nabla,\mathcal{D}]^f{}_{bh}\,\Gamma[\nabla,\mathcal{D}]^h{}_{ac}\,g_{df} - \Gamma[\nabla,\mathcal{D}]^f{}_{ah}\,\Gamma[\nabla,\mathcal{D}]^h{}_{bc}\,g_{df} - \Gamma[\nabla,\mathcal{D}]^f{}_{hc}\,g_{df}\,T^h{}_{ab} - e_b{}^h\,e_c{}^j\,g_{df}\left(\mathcal{D}_a\,\Gamma[\nabla,\mathcal{D}]^f{}_{hj}\right) + e_a{}^h\,e_c{}^j\,g_{df}\left(\mathcal{D}_b\,\Gamma[\nabla,\mathcal{D}]^f{}_{hj}\right)$

In[257]:= `ToBasis[NP] /@%`

Out[257]= $R[\nabla]_{abcd} == \Gamma[\nabla,\mathcal{D}]^f{}_{bh}\,\Gamma[\nabla,\mathcal{D}]^h{}_{ac}\,g_{df} - \Gamma[\nabla,\mathcal{D}]^f{}_{ah}\,\Gamma[\nabla,\mathcal{D}]^h{}_{bc}\,g_{df} - \Gamma[\nabla,\mathcal{D}]^f{}_{hc}\,g_{df}\,T^h{}_{ab} - g_{df}\left(\mathcal{D}_a\,\Gamma[\nabla,\mathcal{D}]^f{}_{bc}\right) + g_{df}\left(\mathcal{D}_b\,\Gamma[\nabla,\mathcal{D}]^f{}_{ac}\right)$

In[258]:= `TraceBasisDummy /@%;`

In[259]:= `Flatten@ComponentArray@%;`

From here we obtain directly the Newman-Penrose equations

In[260]:= `Union[Simplify //@ Union[%]];`

In[261]:= `NPRicci = %`

Out[261]= $\{$True, $\Phi_{01} + 2\,\overline{\alpha}\,\overline{\epsilon} + 2\,\beta\,\overline{\epsilon} + \gamma\,\kappa + \overline{\gamma}\,\kappa + \overline{\kappa}\,\overline{\lambda} + \kappa\,\mu + D\,\overline{\alpha} + D\,\beta ==$

$\Psi_1 + \epsilon\,\overline{\pi} + \overline{\epsilon}\,\overline{\pi} + \overline{\alpha}\,\overline{\rho} + \beta\,\overline{\rho} + \overline{\pi}\,\overline{\rho} + \alpha\,\sigma + \overline{\beta}\,\sigma + \pi\,\sigma + \delta\,\epsilon + \delta\,\overline{\epsilon}\,,$

$\Phi_{10} + 2\,\alpha\,\epsilon + 2\,\overline{\beta}\,\epsilon + \gamma\,\overline{\kappa} + \overline{\gamma}\,\overline{\kappa} + \kappa\,\lambda + \overline{\kappa}\,\overline{\mu} + D\,\alpha + D\,\overline{\beta} ==$

$\overline{\Psi}_1 + \epsilon\,\pi + \overline{\epsilon}\,\pi + \alpha\,\rho + \overline{\beta}\,\rho + \pi\,\rho + \overline{\alpha}\,\overline{\sigma} + \beta\,\overline{\sigma} + \overline{\pi}\,\overline{\sigma} + \overline{\delta}\,\epsilon + \overline{\delta}\,\overline{\epsilon}\,,$

$2\,\Phi_{11} + 2\,(\gamma + \overline{\gamma})\,(\epsilon + \overline{\epsilon}) + \kappa\,\nu + \overline{\kappa}\,\overline{\nu} + D\,\gamma + D\,\overline{\gamma} ==$

$2\,\Lambda + \Psi_2 + \overline{\Psi}_2 + \pi\,\tau + \left(\alpha + \overline{\beta}\right)\,(\overline{\pi} + \tau) + \overline{\pi}\,\overline{\tau} + (\overline{\alpha} + \beta)\,(\pi + \overline{\tau}) + \Delta\,\epsilon + \Delta\,\overline{\epsilon}\,,$

$\Phi_{20} + 3\,\epsilon\,\lambda + \overline{\kappa}\,\nu + \overline{\beta}\,\pi + D\,\lambda == \overline{\epsilon}\,\lambda + \alpha\,\pi + \pi^2 + \lambda\,\rho + \mu\,\overline{\sigma} + \overline{\delta}\,\pi\,,$

$\Phi_{02} + 3\,\overline{\epsilon}\,\overline{\lambda} + \kappa\,\overline{\nu} + \beta\,\overline{\pi} + D\,\overline{\lambda} == \epsilon\,\overline{\lambda} + \overline{\alpha}\,\overline{\pi} + \overline{\pi}^2 + \overline{\lambda}\,\overline{\rho} + \overline{\mu}\,\sigma + \delta\,\overline{\pi}\,,$

$2\,\Lambda + \epsilon\,\mu + \overline{\epsilon}\,\mu + \kappa\,\nu + \overline{\alpha}\,\pi + D\,\mu == \Psi_2 + \beta\,\pi + \pi\,\overline{\pi} + \mu\,\overline{\rho} + \lambda\,\sigma + \delta\,\pi\,,$

$2\,\Lambda + \epsilon\,\overline{\mu} + \overline{\epsilon}\,\overline{\mu} + \overline{\kappa}\,\overline{\nu} + \alpha\,\overline{\pi} + D\,\overline{\mu} == \overline{\Psi}_2 + \overline{\beta}\,\overline{\pi} + \pi\,\overline{\pi} + \overline{\mu}\,\rho + \overline{\lambda}\,\overline{\sigma} + \overline{\delta}\,\overline{\pi}\,,$

$\Phi_{21} + 3\,\epsilon\,\nu + \overline{\epsilon}\,\nu + \overline{\gamma}\,\pi + D\,\nu == \Psi_3 + \gamma\,\pi + \mu\,\pi + \lambda\,\overline{\pi} + \lambda\,\tau + \mu\,\overline{\tau} + \Delta\,\pi\,,$

$\Phi_{12} + \epsilon\,\overline{\nu} + 3\,\overline{\epsilon}\,\overline{\nu} + \gamma\,\overline{\pi} + D\,\overline{\nu} == \overline{\Psi}_3 + \overline{\lambda}\,\pi + \overline{\gamma}\,\overline{\pi} + \overline{\mu}\,\overline{\pi} + \overline{\mu}\,\tau + \overline{\lambda}\,\overline{\tau} + \Delta\,\overline{\pi}\,,$

$\Phi_{00} + 3\,\alpha\,\kappa + \overline{\beta}\,\kappa + \overline{\kappa}\,\tau + D\,\rho == \kappa\,\pi + \epsilon\,\rho + \overline{\epsilon}\,\rho + \rho^2 + \sigma\,\overline{\sigma} + \overline{\delta}\,\kappa\,,$

$\Phi_{00} + 3\,\overline{\alpha}\,\overline{\kappa} + \beta\,\overline{\kappa} + \kappa\,\overline{\tau} + D\,\overline{\rho} == \overline{\kappa}\,\overline{\pi} + \epsilon\,\overline{\rho} + \overline{\epsilon}\,\overline{\rho} + \overline{\rho}^2 + \sigma\,\overline{\sigma} + \delta\,\overline{\kappa}\,,$

$\Phi_{01} + 3\,\gamma\,\kappa + \overline{\gamma}\,\kappa + \overline{\epsilon}\,\tau + D\,\tau == \Psi_1 + \overline{\pi}\,\rho + \pi\,\sigma + \epsilon\,\tau + \rho\,\tau + \sigma\,\overline{\tau} + \Delta\,\kappa\,,$

$\Phi_{10} + \gamma\,\overline{\kappa} + 3\,\overline{\gamma}\,\overline{\kappa} + \epsilon\,\overline{\tau} + D\,\overline{\tau} == \overline{\Psi}_1 + \pi\,\overline{\rho} + \overline{\pi}\,\overline{\sigma} + \overline{\sigma}\,\tau + \overline{\epsilon}\,\overline{\tau} + \overline{\rho}\,\overline{\tau} + \Delta\,\overline{\kappa}\,,$

$\Psi_2 + 2\,\overline{\gamma}\,\overline{\epsilon} + \overline{\kappa}\,\overline{\nu} + \beta\,\pi + \alpha\,\overline{\pi} + \alpha\,\tau + \pi\,\tau + \beta\,\overline{\tau} + D\,\overline{\gamma} + \Delta\,\epsilon ==$

$\overline{\Psi}_2 + 2\,\gamma\,\epsilon + \kappa\,\nu + \overline{\alpha}\,\pi + \overline{\beta}\,\overline{\pi} + \overline{\beta}\,\tau + \overline{\alpha}\,\overline{\tau} + \overline{\pi}\,\overline{\tau} + D\,\gamma + \Delta\,\overline{\epsilon}\,,$

$2\,\Lambda + \Psi_2 + \overline{\Psi}_2 + \pi\,\tau + \left(\alpha + \overline{\beta}\right)\,(\overline{\pi} + \tau) + \overline{\pi}\,\overline{\tau} + (\overline{\alpha} + \beta)\,(\pi + \overline{\tau}) + \Delta\,\epsilon + \Delta\,\overline{\epsilon} ==$

$2\,\Phi_{11} + 2\,(\gamma + \overline{\gamma})\,(\epsilon + \overline{\epsilon}) + \kappa\,\nu + \overline{\kappa}\,\overline{\nu} + D\,\gamma + D\,\overline{\gamma}\,,$

$\Psi_4 + 3\,\gamma\,\lambda + \lambda\,\mu + \lambda\,\overline{\mu} + \nu\,\overline{\tau} + \Delta\,\lambda == \overline{\gamma}\,\lambda + \nu\,\left(3\,\alpha + \overline{\beta} + \pi\right) + \overline{\delta}\,\nu\,,$

$\overline{\Psi}_4 + 3\,\overline{\gamma}\,\overline{\lambda} + \overline{\lambda}\,\mu + \overline{\lambda}\,\overline{\mu} + \overline{\nu}\,\tau + \Delta\,\overline{\lambda} == \gamma\,\overline{\lambda} + \overline{\nu}\,(3\,\overline{\alpha} + \beta + \overline{\pi}) + \delta\,\overline{\nu}\,,$

$\Psi_2 + 2\,\alpha\,\beta + \epsilon\,\overline{\mu} + \overline{\epsilon}\,\overline{\mu} + \gamma\,\overline{\rho} + \overline{\gamma}\,\overline{\rho} + \overline{\mu}\,\overline{\rho} + \lambda\,\sigma + \delta\,\alpha + \delta\,\overline{\beta} ==$

$\overline{\Psi}_2 + 2\,\overline{\alpha}\,\overline{\beta} + \epsilon\,\mu + \overline{\epsilon}\,\mu + \gamma\,\rho + \overline{\gamma}\,\rho + \mu\,\rho + \overline{\lambda}\,\overline{\sigma} + \overline{\delta}\,\overline{\alpha} + \overline{\delta}\,\beta\,,$

$\Phi_{12} + \overline{\Psi}_3 + 2\,\beta\,\gamma + \overline{\beta}\,\overline{\lambda} + \overline{\alpha}\,\mu + \epsilon\,\overline{\nu} + \nu\,\sigma + \overline{\gamma}\,\tau + \overline{\lambda}\,\overline{\tau} + \Delta\,\overline{\alpha} + \delta\,\gamma ==$

$2\,\beta\,\overline{\gamma} + \alpha\,\overline{\lambda} + \beta\,\mu + \overline{\epsilon}\,\overline{\nu} + \overline{\nu}\,\overline{\rho} + \gamma\,\tau + \mu\,\tau + \Delta\,\beta + \delta\,\overline{\gamma}\,,$

$\Phi_{12} + 2\,\overline{\alpha}\,\gamma + 2\,\beta\,\gamma + \epsilon\,\overline{\nu} + \overline{\epsilon}\,\overline{\nu} + \overline{\nu}\,\overline{\rho} + \nu\,\sigma + \delta\,\gamma + \delta\,\overline{\gamma} ==$

$\overline{\Psi}_3 + \alpha\,\overline{\lambda} + \overline{\beta}\,\overline{\lambda} + \overline{\alpha}\,\mu + \beta\,\mu + \gamma\,\tau + \overline{\gamma}\,\tau + \mu\,\tau + \overline{\lambda}\,\overline{\tau} + \Delta\,\overline{\alpha} + \Delta\,\beta\,,$

$\Phi_{01} + \Psi_1 + 2\,\overline{\alpha}\,\overline{\epsilon} + \overline{\gamma}\,\kappa + \overline{\kappa}\,\overline{\lambda} + \epsilon\,\overline{\pi} + \beta\,\overline{\rho} + \alpha\,\sigma + \pi\,\sigma + D\,\overline{\alpha} + \delta\,\epsilon ==$

$2\,\overline{\alpha}\,\epsilon + \gamma\,\kappa + \kappa\,\mu + \overline{\epsilon}\,\overline{\pi} + \overline{\alpha}\,\overline{\rho} + \overline{\pi}\,\overline{\rho} + \overline{\beta}\,\sigma + D\,\beta + \delta\,\overline{\epsilon}\,,$

$\Psi_0 + \kappa\,\overline{\pi} + 3\,\epsilon\,\sigma + \rho\,\sigma + \overline{\rho}\,\sigma + \delta\,\kappa == \overline{\alpha}\,\kappa + 3\,\beta\,\kappa + \overline{\epsilon}\,\sigma + \kappa\,\tau + D\,\sigma\,,$

$\Phi_{21} + \Psi_3 + 3\,\beta\,\lambda + \overline{\mu}\,\pi + \nu\,\overline{\rho} + \delta\,\lambda == \overline{\alpha}\,\lambda + \alpha\,\mu + \overline{\beta}\,\mu + \mu\,\pi + \nu\,\rho + \overline{\delta}\,\mu\,,$

$\Phi_{22} + \overline{\alpha}\,\nu + 3\,\beta\,\nu + \overline{\nu}\,\pi + \delta\,\nu == \lambda\,\overline{\lambda} + \gamma\,\mu + \overline{\gamma}\,\mu + \mu^2 + \nu\,\tau + \Delta\,\mu\,,$

$\gamma\,\overline{\lambda} + \overline{\nu}\,(3\,\overline{\alpha} + \beta + \overline{\pi}) + \delta\,\overline{\nu} == \overline{\Psi}_4 + 3\,\overline{\gamma}\,\overline{\lambda} + \overline{\lambda}\,\mu + \overline{\lambda}\,\overline{\mu} + \overline{\nu}\,\tau + \Delta\,\overline{\lambda}\,,$

$\Phi_{01} + \Psi_1 + \kappa\,\overline{\mu} + 3\,\alpha\,\sigma + \overline{\rho}\,\tau + \delta\,\rho == \kappa\,\mu + \overline{\alpha}\,\rho + \beta\,\rho + \overline{\beta}\,\sigma + \rho\,\tau + \overline{\delta}\,\sigma\,,$

$\Phi_{02} + \kappa\,\overline{\nu} + 3\,\gamma\,\sigma + \overline{\alpha}\,\tau + \delta\,\tau == \overline{\lambda}\,\rho + \overline{\gamma}\,\sigma + \mu\,\sigma + \beta\,\tau + \tau^2 + \Delta\,\sigma\,,$

$2\,\Lambda + \overline{\kappa}\,\overline{\nu} + \gamma\,\overline{\rho} + \overline{\gamma}\,\overline{\rho} + \beta\,\overline{\tau} + \delta\,\overline{\tau} == \overline{\Psi}_2 + \mu\,\overline{\rho} + \overline{\lambda}\,\overline{\sigma} + \overline{\alpha}\,\overline{\tau} + \tau\,\overline{\tau} + \Delta\,\overline{\rho}\,,$

$2\,\Lambda + 2\,\Phi_{11} + \Psi_2 + \overline{\Psi}_2 + 2\,\alpha\,\beta + 2\,\overline{\alpha}\,\overline{\beta} + \overline{\epsilon}\,\mu + \epsilon\,\overline{\mu} + \overline{\gamma}\,\rho + \gamma\,\overline{\rho} + \lambda\,\sigma + \overline{\lambda}\,\overline{\sigma} + \delta\,\alpha + \overline{\delta}\,\overline{\alpha} ==$

$2\,\alpha\,\overline{\alpha} + 2\,\beta\,\overline{\beta} + \epsilon\,\mu + \overline{\epsilon}\,\overline{\mu} + \gamma\,\rho + \mu\,\rho + \overline{\gamma}\,\overline{\rho} + \overline{\mu}\,\overline{\rho} + \delta\,\overline{\beta} + \overline{\delta}\,\beta\,,$

$\Phi_{21} + \Psi_3 + 2\,\overline{\beta}\,\overline{\gamma} + \beta\,\lambda + \alpha\,\overline{\mu} + \overline{\epsilon}\,\nu + \overline{\nu}\,\overline{\sigma} + \lambda\,\tau + \gamma\,\overline{\tau} + \Delta\,\alpha + \overline{\delta}\,\overline{\gamma} ==$

$$2\,\overline{\beta}\,\gamma + \overline{\alpha}\,\lambda + \overline{\beta}\,\overline{\mu} + \epsilon\,\nu + \nu\,\rho + \overline{\gamma}\,\overline{\tau} + \overline{\mu}\,\overline{\tau} + \Delta\,\overline{\beta} + \delta\,\gamma\,,$$

$$\Phi_{21} + 2\,\alpha\,\overline{\gamma} + 2\,\overline{\beta}\,\overline{\gamma} + \epsilon\,\nu + \overline{\epsilon}\,\nu + \nu\,\rho + \overline{\nu}\,\overline{\sigma} + \overline{\delta}\,\gamma + \overline{\delta}\,\overline{\gamma} ==$$
$$\Psi_3 + \overline{\alpha}\,\lambda + \beta\,\lambda + \alpha\,\overline{\mu} + \overline{\beta}\,\overline{\mu} + \lambda\,\tau + \gamma\,\overline{\tau} + \overline{\gamma}\,\overline{\tau} + \overline{\mu}\,\overline{\tau} + \Delta\,\alpha + \Delta\,\overline{\beta}\,,$$

$$\Phi_{10} + \overline{\Psi_1} + 2\,\alpha\,\epsilon + \gamma\,\overline{\kappa} + \kappa\,\lambda + \overline{\epsilon}\,\pi + \overline{\beta}\,\rho + \overline{\alpha}\,\overline{\sigma} + \overline{\pi}\,\overline{\sigma} + D\,\alpha + \overline{\delta}\,\overline{\epsilon} ==$$
$$2\,\alpha\,\overline{\epsilon} + \overline{\gamma}\,\overline{\kappa} + \overline{\kappa}\,\overline{\mu} + \epsilon\,\pi + \alpha\,\rho + \pi\,\rho + \beta\,\overline{\sigma} + D\,\overline{\beta} + \overline{\delta}\,\epsilon\,,$$

$$\overline{\Psi_0} + \overline{\kappa}\,\pi + 3\,\overline{\epsilon}\,\overline{\sigma} + \rho\,\overline{\sigma} + \overline{\rho}\,\overline{\sigma} + \overline{\delta}\,\overline{\kappa} == \alpha\,\overline{\kappa} + 3\,\overline{\beta}\,\overline{\kappa} + \epsilon\,\overline{\sigma} + \overline{\kappa}\,\overline{\tau} + D\,\overline{\sigma}\,,$$

$$\Phi_{12} + \overline{\Psi_3} + 3\,\overline{\beta}\,\overline{\lambda} + \mu\,\overline{\pi} + \overline{\nu}\,\rho + \overline{\delta}\,\overline{\lambda} == \alpha\,\overline{\lambda} + \overline{\alpha}\,\overline{\mu} + \beta\,\overline{\mu} + \overline{\mu}\,\overline{\pi} + \overline{\nu}\,\overline{\rho} + \delta\,\overline{\mu}\,,$$

$$\overline{\gamma}\,\lambda + \nu\,\left(3\,\alpha + \overline{\beta} + \pi\right) + \overline{\delta}\,\nu == \Psi_4 + 3\,\gamma\,\lambda + \lambda\,\mu + \lambda\,\overline{\mu} + \nu\,\overline{\tau} + \Delta\,\lambda\,,$$

$$\Phi_{22} + \alpha\,\overline{\nu} + 3\,\overline{\beta}\,\overline{\nu} + \nu\,\overline{\pi} + \overline{\delta}\,\overline{\nu} == \lambda\,\overline{\lambda} + \gamma\,\overline{\mu} + \overline{\gamma}\,\overline{\mu} + \overline{\mu}^2 + \overline{\nu}\,\overline{\tau} + \Delta\,\overline{\mu}\,,$$

$$\Phi_{10} + \overline{\Psi_1} + \overline{\kappa}\,\mu + 3\,\overline{\alpha}\,\overline{\sigma} + \rho\,\overline{\tau} + \overline{\delta}\,\overline{\rho} == \overline{\kappa}\,\overline{\mu} + \alpha\,\overline{\rho} + \overline{\beta}\,\overline{\rho} + \beta\,\overline{\sigma} + \overline{\rho}\,\overline{\tau} + \delta\,\overline{\sigma}\,,$$

$$2\,\Lambda + \kappa\,\nu + \gamma\,\rho + \overline{\gamma}\,\rho + \overline{\beta}\,\tau + \overline{\delta}\,\tau == \Psi_2 + \overline{\mu}\,\rho + \lambda\,\sigma + \alpha\,\tau + \tau\,\overline{\tau} + \Delta\,\rho\,,$$

$$\Phi_{20} + \overline{\kappa}\,\nu + 3\,\overline{\gamma}\,\overline{\sigma} + \alpha\,\overline{\tau} + \overline{\delta}\,\overline{\tau} == \lambda\,\overline{\rho} + \gamma\,\overline{\sigma} + \overline{\mu}\,\overline{\sigma} + \overline{\beta}\,\overline{\tau} + \overline{\tau}^2 + \Delta\,\overline{\sigma}\,\big\}$$

Not all the equations in this bunch are independent. To extract the independent set of equations we just eliminate a certain set of variables. The variables which we are going to eliminate are

In[262]:= **Dagger /@**
       Flatten@Outer[PDNP[#1]@#2 &, {{1, -NP}, {2, -NP}, {3, -NP}, {4, -NP}}, NPSpinCoefficients, 1]

Out[262]= $\big\{D\,\overline{\alpha}\,,\,D\,\overline{\beta}\,,\,D\,\overline{\gamma}\,,\,D\,\overline{\epsilon}\,,\,D\,\overline{\kappa}\,,\,D\,\overline{\lambda}\,,\,D\,\overline{\mu}\,,\,D\,\overline{\nu}\,,\,D\,\overline{\pi}\,,\,D\,\overline{\rho}\,,\,D\,\overline{\sigma}\,,\,D\,\overline{\tau}\,,\,\Delta\,\overline{\alpha}\,,\,\Delta\,\overline{\beta}\,,\,\Delta\,\overline{\gamma}\,,\,\Delta\,\overline{\epsilon}\,,$
$\Delta\,\overline{\kappa}\,,\,\Delta\,\overline{\lambda}\,,\,\Delta\,\overline{\mu}\,,\,\Delta\,\overline{\nu}\,,\,\Delta\,\overline{\pi}\,,\,\Delta\,\overline{\rho}\,,\,\Delta\,\overline{\sigma}\,,\,\Delta\,\overline{\tau}\,,\,\overline{\delta}\,\overline{\alpha}\,,\,\overline{\delta}\,\overline{\beta}\,,\,\overline{\delta}\,\overline{\gamma}\,,\,\overline{\delta}\,\overline{\epsilon}\,,\,\overline{\delta}\,\overline{\kappa}\,,\,\overline{\delta}\,\overline{\lambda}\,,\,\overline{\delta}\,\overline{\mu}\,,\,\overline{\delta}\,\overline{\nu}\,,$
$\overline{\delta}\,\overline{\pi}\,,\,\overline{\delta}\,\overline{\rho}\,,\,\overline{\delta}\,\overline{\sigma}\,,\,\overline{\delta}\,\overline{\tau}\,,\,\delta\,\overline{\alpha}\,,\,\delta\,\overline{\beta}\,,\,\delta\,\overline{\gamma}\,,\,\delta\,\overline{\epsilon}\,,\,\delta\,\overline{\kappa}\,,\,\delta\,\overline{\lambda}\,,\,\delta\,\overline{\mu}\,,\,\delta\,\overline{\nu}\,,\,\delta\,\overline{\pi}\,,\,\delta\,\overline{\rho}\,,\,\delta\,\overline{\sigma}\,,\,\delta\,\overline{\tau}\big\}$

In[263]:= **Eliminate[NPRicci, %] // Simplify;**

In[264]:= **List @@ %;**

In[265]:= **Solve[%, Union@Cases[%, PDNP[_]@_, Infinity]] // Flatten // Simplify**

   Solve::svars : Equations may not give solutions for all "solve" variables. ≫

Out[265]= $\big\{D\,\alpha \to -\Phi_{10} - 2\,\alpha\,\epsilon - \overline{\beta}\,\epsilon + \alpha\,\overline{\epsilon} - \gamma\,\overline{\kappa} - \kappa\,\lambda + \epsilon\,\pi + \alpha\,\rho + \pi\,\rho + \beta\,\overline{\sigma} + \overline{\delta}\,\epsilon\,,$
$D\,\beta \to \Psi_1 - \overline{\alpha}\,\epsilon - \beta\,\overline{\epsilon} - \gamma\,\kappa - \kappa\,\mu + \epsilon\,\overline{\pi} + \beta\,\overline{\rho} + \alpha\,\sigma + \pi\,\sigma + \delta\,\epsilon\,,$
$D\,\gamma \to \Lambda - \Phi_{11} + \Psi_2 - 2\,\gamma\,\epsilon - \overline{\gamma}\,\epsilon - \gamma\,\overline{\epsilon} - \kappa\,\nu + \beta\,\pi + \alpha\,\overline{\pi} + \alpha\,\tau + \pi\,\tau + \beta\,\overline{\tau} + \Delta\,\epsilon\,,$
$D\,\lambda \to -\Phi_{20} - 3\,\epsilon\,\lambda + \overline{\epsilon}\,\lambda - \overline{\kappa}\,\nu + \alpha\,\pi - \overline{\beta}\,\pi + \pi^2 + \lambda\,\rho + \mu\,\overline{\sigma} + \overline{\delta}\,\pi\,,$
$D\,\mu \to -2\,\Lambda + \Psi_2 - \epsilon\,\mu - \overline{\epsilon}\,\mu - \kappa\,\nu - \overline{\alpha}\,\pi + \beta\,\pi + \pi\,\overline{\pi} + \mu\,\overline{\rho} + \lambda\,\sigma + \delta\,\pi\,,$
$D\,\nu \to -\Phi_{21} + \Psi_3 - 3\,\epsilon\,\nu - \overline{\epsilon}\,\nu + \gamma\,\pi - \overline{\gamma}\,\pi + \mu\,\pi + \lambda\,\overline{\pi} + \lambda\,\tau + \mu\,\overline{\tau} + \Delta\,\pi\,,$
$D\,\rho \to -\Phi_{00} - 3\,\alpha\,\kappa - \overline{\beta}\,\kappa + \kappa\,\pi + \epsilon\,\rho + \overline{\epsilon}\,\rho + \rho^2 + \sigma\,\overline{\sigma} - \overline{\kappa}\,\tau + \overline{\delta}\,\kappa\,,$
$D\,\sigma \to \Psi_0 - \overline{\alpha}\,\kappa - 3\,\beta\,\kappa + \kappa\,\overline{\pi} + 3\,\epsilon\,\sigma - \overline{\epsilon}\,\sigma + \rho\,\sigma + \overline{\rho}\,\sigma - \kappa\,\tau + \delta\,\kappa\,,$
$D\,\tau \to -\Phi_{01} + \Psi_1 - 3\,\gamma\,\kappa - \overline{\gamma}\,\kappa + \overline{\pi}\,\rho + \pi\,\sigma + \epsilon\,\tau - \overline{\epsilon}\,\tau + \rho\,\tau + \sigma\,\overline{\tau} + \Delta\,\kappa\,,$
$\Delta\,\alpha \to -\overline{\Psi_3} + \overline{\beta}\,\gamma + \alpha\,\overline{\gamma} - \beta\,\lambda - \alpha\,\overline{\mu} + \epsilon\,\nu + \nu\,\rho - \lambda\,\tau - \gamma\,\overline{\tau} + \overline{\delta}\,\gamma\,,$
$\Delta\,\beta \to \Phi_{12} + \overline{\alpha}\,\gamma + 2\,\beta\,\gamma - \beta\,\overline{\gamma} - \alpha\,\overline{\lambda} - \beta\,\mu + \epsilon\,\overline{\nu} + \nu\,\sigma - \gamma\,\tau - \mu\,\tau + \delta\,\gamma\,,$
$\Delta\,\lambda \to -\overline{\Psi_4} - 3\,\gamma\,\lambda + \overline{\gamma}\,\lambda - \lambda\,\mu - \lambda\,\overline{\mu} + 3\,\alpha\,\nu + \overline{\beta}\,\nu + \nu\,\pi - \nu\,\overline{\tau} + \overline{\delta}\,\nu\,,$
$\Delta\,\mu \to \Phi_{22} - \lambda\,\overline{\lambda} - \gamma\,\mu - \overline{\gamma}\,\mu - \mu^2 + \overline{\alpha}\,\nu + 3\,\beta\,\nu + \overline{\nu}\,\pi - \nu\,\tau + \delta\,\nu\,,$
$\Delta\,\rho \to 2\,\Lambda - \Psi_2 + \kappa\,\nu + \gamma\,\rho + \overline{\gamma}\,\rho - \overline{\mu}\,\rho - \lambda\,\sigma - \alpha\,\tau + \overline{\beta}\,\tau - \tau\,\overline{\tau} + \overline{\delta}\,\tau\,,$
$\Delta\,\sigma \to \Phi_{02} + \kappa\,\overline{\nu} - \overline{\lambda}\,\rho + 3\,\gamma\,\sigma - \overline{\gamma}\,\sigma - \mu\,\sigma + \overline{\alpha}\,\tau - \beta\,\tau - \tau^2 + \delta\,\tau\,,$
$\delta\,\alpha \to -\Lambda - \Phi_{11} - \Psi_2 + \alpha\,\overline{\alpha} - 2\,\alpha\,\beta + \beta\,\overline{\beta} + \epsilon\,\mu - \epsilon\,\overline{\mu} + \gamma\,\rho + \mu\,\rho - \gamma\,\overline{\rho} - \lambda\,\sigma + \overline{\delta}\,\beta\,,$
$\delta\,\lambda \to -\Phi_{21} - \Psi_3 + \overline{\alpha}\,\lambda - 3\,\beta\,\lambda + \alpha\,\mu + \overline{\beta}\,\mu + \mu\,\pi - \overline{\mu}\,\pi + \nu\,\rho - \nu\,\overline{\rho} + \overline{\delta}\,\mu\,,$
$\delta\,\rho \to -\Phi_{01} - \Psi_1 + \kappa\,\mu - \kappa\,\overline{\mu} + \overline{\alpha}\,\rho + \beta\,\rho - 3\,\alpha\,\sigma + \overline{\beta}\,\sigma + \rho\,\tau - \overline{\rho}\,\tau + \overline{\delta}\,\sigma\big\}$

In[266]:= **NPRicci = (% /. Rule -> Equal);**

In[267]:= **Map[FullSimplify, NPRicci, {2}]**

Out[267]= $\{ D\,\alpha \;==\; -\Phi_{10} \;-\overline{\beta}\;\epsilon \;-\gamma\;\overline{\kappa} \;-\kappa\;\lambda \;+\pi\;(\epsilon \;+\rho\;) \;+\alpha\;(-2\,\epsilon \;+\overline{\epsilon} \;+\rho\;) \;+\beta\;\overline{\sigma} \;+\overline{\delta}\,\epsilon \;,$

$\quad D\,\beta \;==\; \Psi_1 \;-\kappa\;(\gamma \;+\mu\;) \;+\epsilon\;(-\overline{\alpha} \;+\overline{\pi}\;) \;+\beta\;(-\overline{\epsilon} \;+\overline{\rho}\;) \;+(\alpha \;+\pi\;)\,\sigma \;+\delta\,\epsilon \;,$

$\quad D\,\gamma \;==\; \Lambda \;-\Phi_{11} \;+\Psi_2 \;-(2\,\gamma \;+\overline{\gamma}\;)\,\epsilon \;-\gamma\;\overline{\epsilon} \;-\kappa\;\nu \;+\pi\;(\beta \;+\tau\;) \;+\alpha\;(\overline{\pi} \;+\tau\;) \;+\beta\;\overline{\tau} \;+\Delta\,\epsilon \;,$

$\quad D\,\lambda \;==\; -\Phi_{20} \;-\overline{\kappa}\;\nu \;+\pi\;(\alpha \;-\overline{\beta} \;+\pi\;) \;+\lambda\;(-3\,\epsilon \;+\overline{\epsilon} \;+\rho\;) \;+\mu\;\overline{\sigma} \;+\overline{\delta}\,\pi \;,$

$\quad D\,\mu \;==\; -2\,\Lambda \;+\Psi_2 \;-\kappa\;\nu \;+\pi\;(-\overline{\alpha} \;+\beta \;+\overline{\pi}\;) \;-\mu\;(\epsilon \;+\overline{\epsilon} \;-\overline{\rho}\;) \;+\lambda\;\sigma \;+\delta\,\pi \;,$

$\quad D\,\nu \;==\; -\Phi_{21} \;+\Psi_3 \;-(3\,\epsilon \;+\overline{\epsilon}\;)\,\nu \;+(\gamma \;-\overline{\gamma} \;+\mu\;)\,\pi \;+\lambda\;(\overline{\pi} \;+\tau\;) \;+\mu\;\overline{\tau} \;+\Delta\,\pi \;,$

$\quad D\,\rho \;==\; -\Phi_{00} \;+\kappa\;(-3\,\alpha \;-\overline{\beta} \;+\pi\;) \;+\rho\;(\epsilon \;+\overline{\epsilon} \;+\rho\;) \;+\sigma\;\overline{\sigma} \;-\overline{\kappa}\;\tau \;+\overline{\delta}\,\kappa \;,$

$\quad D\,\sigma \;==\; \Psi_0 \;+(3\,\epsilon \;-\overline{\epsilon} \;+\rho \;+\overline{\rho}\;)\,\sigma \;-\kappa\;(\overline{\alpha} \;+3\,\beta \;-\overline{\pi} \;+\tau\;) \;+\delta\,\kappa \;,$

$\quad D\,\tau \;==\; -\Phi_{01} \;+\Psi_1 \;-(3\,\gamma \;+\overline{\gamma}\;)\,\kappa \;+\overline{\pi}\;\rho \;+(\epsilon \;-\overline{\epsilon} \;+\rho\;)\,\tau \;+\sigma\;(\pi \;+\overline{\tau}\;) \;+\Delta\,\kappa \;,$

$\quad \Delta\,\alpha \;==\; -\Psi_3 \;+\alpha\;(\overline{\gamma} \;-\overline{\mu}\;) \;+\nu\;(\epsilon \;+\rho\;) \;-\lambda\;(\beta \;+\tau\;) \;+\gamma\;(\overline{\beta} \;-\overline{\tau}\;) \;+\overline{\delta}\,\gamma \;,$

$\quad \Delta\,\beta \;==\; \Phi_{12} \;+\overline{\alpha}\;\gamma \;+2\,\beta\;\gamma \;-\beta\;\overline{\gamma} \;-\alpha\;\overline{\lambda} \;-\beta\;\mu \;+\epsilon\;\overline{\nu} \;+\nu\;\sigma \;-(\gamma \;+\mu\;)\,\tau \;+\delta\,\gamma \;,$

$\quad \Delta\,\lambda \;==\; -\Psi_4 \;-\lambda\;(3\,\gamma \;-\overline{\gamma} \;+\mu \;+\overline{\mu}\;) \;+\nu\;(3\,\alpha \;+\overline{\beta} \;+\pi \;-\overline{\tau}\;) \;+\overline{\delta}\,\nu \;,$

$\quad \Delta\,\mu \;==\; \Phi_{22} \;-\lambda\;\overline{\lambda} \;-\mu\;(\gamma \;+\overline{\gamma} \;+\mu\;) \;+\overline{\nu}\;\pi \;+\nu\;(\overline{\alpha} \;+3\,\beta \;-\tau\;) \;+\delta\,\nu \;,$

$\quad \Delta\,\rho \;==\; 2\,\Lambda \;-\Psi_2 \;+\kappa\;\nu \;+(\gamma \;+\overline{\gamma} \;-\overline{\mu}\;)\,\rho \;-\lambda\;\sigma \;-\tau\;(\alpha \;-\overline{\beta} \;+\overline{\tau}\;) \;+\overline{\delta}\,\tau \;,$

$\quad \Delta\,\sigma \;==\; \Phi_{02} \;+\kappa\;\overline{\nu} \;-\overline{\lambda}\;\rho \;+3\,\gamma\;\sigma \;-(\overline{\gamma} \;+\mu\;)\,\sigma \;+(\overline{\alpha} \;-\beta \;-\tau\;)\,\tau \;+\delta\,\tau \;,$

$\quad \delta\,\alpha \;==\; -\Lambda \;-\Phi_{11} \;-\Psi_2 \;+\alpha\;(\overline{\alpha} \;-2\,\beta\;) \;+\beta\;\overline{\beta} \;+\epsilon\;(\mu \;-\overline{\mu}\;) \;+(\gamma \;+\mu\;)\,\rho \;-\gamma\;\overline{\rho} \;-\lambda\;\sigma \;+\overline{\delta}\,\beta \;,$

$\quad \delta\,\lambda \;==\; -\Phi_{21} \;-\Psi_3 \;+(\overline{\alpha} \;-3\,\beta\;)\,\lambda \;+(\alpha \;+\overline{\beta}\;)\,\mu \;+(\mu \;-\overline{\mu}\;)\,\pi \;+\nu\;(\rho \;-\overline{\rho}\;) \;+\overline{\delta}\,\mu \;,$

$\quad \delta\,\rho \;==\; -\Phi_{01} \;-\Psi_1 \;+\kappa\;(\mu \;-\overline{\mu}\;) \;+(\overline{\alpha} \;+\beta\;)\,\rho \;+(-3\,\alpha \;+\overline{\beta}\;)\,\sigma \;+(\rho \;-\overline{\rho}\;)\,\tau \;+\overline{\delta}\,\sigma \;\}$

We may need the complex conjugate set in future computations.

In[268]:= **NPRicci† = Map[Dagger, NPRicci, {2}]**

Out[268]= $\{ D\,\overline{\alpha} \;==\; -\Phi_{01} \;+\overline{\alpha}\;\epsilon \;-2\,\overline{\alpha}\;\overline{\epsilon} \;-\beta\;\overline{\epsilon} \;-\overline{\gamma}\;\kappa \;-\overline{\kappa}\;\overline{\lambda} \;+\overline{\epsilon}\;\pi \;+\overline{\alpha}\;\overline{\rho} \;+\pi\;\overline{\rho} \;+\overline{\beta}\;\sigma \;+\delta\,\overline{\epsilon} \;,$

$\quad D\,\overline{\beta} \;==\; \overline{\Psi}_1 \;-\overline{\beta}\;\epsilon \;-\alpha\;\overline{\epsilon} \;-\overline{\gamma}\;\kappa \;-\overline{\kappa}\;\overline{\mu} \;+\overline{\epsilon}\;\pi \;+\overline{\beta}\;\rho \;+\overline{\alpha}\;\overline{\sigma} \;+\pi\;\overline{\sigma} \;+\overline{\delta}\,\overline{\epsilon} \;,$

$\quad D\,\overline{\gamma} \;==\; \Lambda \;-\Phi_{11} \;+\overline{\Psi}_2 \;-\overline{\gamma}\;\epsilon \;-\gamma\;\overline{\epsilon} \;-2\,\overline{\gamma}\;\overline{\epsilon} \;-\overline{\kappa}\;\overline{\nu} \;+\overline{\alpha}\;\pi \;+\overline{\beta}\;\overline{\pi} \;+\overline{\beta}\;\tau \;+\overline{\alpha}\;\overline{\tau} \;+\pi\;\overline{\tau} \;+\Delta\,\overline{\epsilon} \;,$

$\quad D\,\overline{\lambda} \;==\; -\Phi_{02} \;+\epsilon\;\overline{\lambda} \;-3\,\overline{\epsilon}\;\overline{\lambda} \;-\kappa\;\overline{\nu} \;+\overline{\alpha}\;\overline{\pi} \;-\beta\;\overline{\pi} \;+\overline{\pi}^2 \;+\overline{\lambda}\;\overline{\rho} \;+\overline{\mu}\;\sigma \;+\delta\,\overline{\pi} \;,$

$\quad D\,\overline{\mu} \;==\; -2\,\Lambda \;+\overline{\Psi}_2 \;-\epsilon\;\overline{\mu} \;-\overline{\epsilon}\;\overline{\mu} \;-\overline{\kappa}\;\overline{\nu} \;-\alpha\;\overline{\pi} \;+\overline{\beta}\;\overline{\pi} \;+\pi\;\overline{\pi} \;+\overline{\mu}\;\rho \;+\overline{\lambda}\;\overline{\sigma} \;+\overline{\delta}\,\overline{\pi} \;,$

$\quad D\,\overline{\nu} \;==\; -\Phi_{12} \;+\overline{\Psi}_3 \;-\epsilon\;\overline{\nu} \;-3\,\overline{\epsilon}\;\overline{\nu} \;+\overline{\lambda}\;\pi \;-\gamma\;\overline{\pi} \;+\overline{\gamma}\;\overline{\pi} \;+\overline{\mu}\;\overline{\pi} \;+\overline{\mu}\;\tau \;+\overline{\lambda}\;\overline{\tau} \;+\Delta\,\overline{\pi} \;,$

$\quad D\,\overline{\rho} \;==\; -\Phi_{00} \;-3\,\overline{\alpha}\;\overline{\kappa} \;-\beta\;\overline{\kappa} \;+\overline{\kappa}\;\pi \;+\epsilon\;\overline{\rho} \;+\overline{\epsilon}\;\overline{\rho} \;+\overline{\rho}^2 \;+\sigma\;\overline{\sigma} \;-\kappa\;\overline{\tau} \;+\delta\,\overline{\kappa} \;,$

$\quad D\,\overline{\sigma} \;==\; \overline{\Psi}_0 \;-\alpha\;\overline{\kappa} \;-3\,\overline{\beta}\;\overline{\kappa} \;+\overline{\kappa}\;\pi \;-\epsilon\;\overline{\sigma} \;+3\,\overline{\epsilon}\;\overline{\sigma} \;+\rho\;\overline{\sigma} \;+\overline{\rho}\;\overline{\sigma} \;-\overline{\kappa}\;\overline{\tau} \;+\overline{\delta}\,\overline{\kappa} \;,$

$\quad D\,\overline{\tau} \;==\; -\Phi_{10} \;+\overline{\Psi}_1 \;-\gamma\;\overline{\kappa} \;-3\,\overline{\gamma}\;\overline{\kappa} \;+\pi\;\overline{\rho} \;+\overline{\pi}\;\overline{\sigma} \;+\overline{\sigma}\;\tau \;-\epsilon\;\overline{\tau} \;+\overline{\epsilon}\;\overline{\tau} \;+\overline{\rho}\;\overline{\tau} \;+\Delta\,\overline{\kappa} \;,$

$\quad \Delta\,\overline{\alpha} \;==\; -\overline{\Psi}_3 \;+\overline{\alpha}\;\gamma \;+\beta\;\overline{\gamma} \;-\overline{\beta}\;\overline{\lambda} \;-\overline{\alpha}\;\mu \;+\overline{\epsilon}\;\overline{\nu} \;+\overline{\nu}\;\overline{\rho} \;-\overline{\gamma}\;\tau \;-\overline{\lambda}\;\overline{\tau} \;+\delta\,\overline{\gamma} \;,$

$\quad \Delta\,\overline{\beta} \;==\; \Phi_{21} \;-\overline{\beta}\;\gamma \;+\alpha\;\overline{\gamma} \;+2\,\overline{\beta}\;\overline{\gamma} \;-\overline{\alpha}\;\lambda \;-\overline{\beta}\;\overline{\mu} \;+\overline{\epsilon}\;\nu \;+\overline{\nu}\;\overline{\sigma} \;-\overline{\gamma}\;\overline{\tau} \;-\overline{\mu}\;\overline{\tau} \;+\overline{\delta}\,\overline{\gamma} \;,$

$\quad \Delta\,\overline{\lambda} \;==\; -\overline{\Psi}_4 \;+\gamma\;\overline{\lambda} \;-3\,\overline{\gamma}\;\overline{\lambda} \;-\overline{\lambda}\;\mu \;-\overline{\lambda}\;\overline{\mu} \;+3\,\overline{\alpha}\;\overline{\nu} \;+\beta\;\overline{\nu} \;+\overline{\nu}\;\overline{\pi} \;-\overline{\nu}\;\tau \;+\delta\,\overline{\nu} \;,$

$\quad \Delta\,\overline{\mu} \;==\; \Phi_{22} \;-\lambda\;\overline{\lambda} \;-\gamma\;\overline{\mu} \;-\overline{\gamma}\;\overline{\mu} \;-\overline{\mu}^2 \;+\alpha\;\overline{\nu} \;+3\,\overline{\beta}\;\overline{\nu} \;+\nu\;\overline{\pi} \;-\overline{\nu}\;\overline{\tau} \;+\overline{\delta}\,\overline{\nu} \;,$

$\quad \Delta\,\overline{\rho} \;==\; 2\,\Lambda \;-\overline{\Psi}_2 \;+\overline{\kappa}\;\overline{\nu} \;+\gamma\;\overline{\rho} \;+\overline{\gamma}\;\overline{\rho} \;-\mu\;\overline{\rho} \;-\overline{\lambda}\;\overline{\sigma} \;-\overline{\alpha}\;\overline{\tau} \;+\beta\;\overline{\tau} \;-\tau\;\overline{\tau} \;+\delta\,\overline{\tau} \;,$

$\quad \Delta\,\overline{\sigma} \;==\; \Phi_{20} \;+\overline{\kappa}\;\nu \;-\lambda\;\overline{\rho} \;-\gamma\;\overline{\sigma} \;+3\,\overline{\gamma}\;\overline{\sigma} \;-\overline{\mu}\;\overline{\sigma} \;+\alpha\;\overline{\tau} \;-\overline{\beta}\;\overline{\tau} \;-\overline{\tau}^2 \;+\overline{\delta}\,\overline{\tau} \;,$

$\quad \overline{\delta}\,\overline{\alpha} \;==\; -\Lambda \;-\Phi_{11} \;-\overline{\Psi}_2 \;+\alpha\;\overline{\alpha} \;-2\,\overline{\alpha}\;\overline{\beta} \;+\beta\;\overline{\beta} \;-\overline{\epsilon}\;\mu \;+\overline{\epsilon}\;\overline{\mu} \;-\overline{\gamma}\;\rho \;+\overline{\gamma}\;\overline{\rho} \;+\overline{\mu}\;\overline{\rho} \;-\overline{\lambda}\;\overline{\sigma} \;+\delta\,\overline{\beta} \;,$

$\quad \overline{\delta}\,\overline{\lambda} \;==\; -\Phi_{12} \;-\overline{\Psi}_3 \;+\alpha\;\overline{\lambda} \;-3\,\overline{\beta}\;\overline{\lambda} \;+\overline{\alpha}\;\overline{\mu} \;+\beta\;\overline{\mu} \;-\mu\;\overline{\pi} \;+\overline{\mu}\;\overline{\pi} \;-\overline{\nu}\;\rho \;+\overline{\nu}\;\overline{\rho} \;+\delta\,\overline{\mu} \;,$

$\quad \overline{\delta}\,\overline{\rho} \;==\; -\Phi_{10} \;-\overline{\Psi}_1 \;-\overline{\kappa}\;\mu \;+\overline{\kappa}\;\overline{\mu} \;+\alpha\;\overline{\rho} \;+\overline{\beta}\;\overline{\rho} \;-3\,\overline{\alpha}\;\overline{\sigma} \;+\beta\;\overline{\sigma} \;-\rho\;\overline{\tau} \;+\overline{\rho}\;\overline{\tau} \;+\delta\,\overline{\sigma} \;\}$

In[269]:= **NPRicci >> "~/calculations/NP5D/NP4D"**

OpenWrite::noopen :
 Cannot open /tmp_mnt/netapp/users_home4/garcia/calculations/NP5D/NP4D. ≫

Out[269]= $\{ D\,\alpha\ ==\ -\Phi_{10}\ -2\,\alpha\ \epsilon\ -\overline{\beta}\ \epsilon\ +\alpha\ \overline{\epsilon}\ -\gamma\ \overline{\kappa}\ -\kappa\ \lambda\ +\epsilon\ \pi\ +\alpha\ \rho\ +\pi\ \rho\ +\beta\ \overline{\sigma}\ +\overline{\delta}\,\epsilon\ ,$

$D\,\beta\ ==\ \Psi_1\ -\overline{\alpha}\ \epsilon\ -\beta\ \overline{\epsilon}\ -\gamma\ \kappa\ -\kappa\ \mu\ +\epsilon\ \overline{\pi}\ +\beta\ \overline{\rho}\ +\alpha\ \sigma\ +\pi\ \sigma\ +\delta\,\epsilon\ ,$

$D\,\gamma\ ==\ \Lambda\ -\Phi_{11}\ +\Psi_2\ -2\,\gamma\ \epsilon\ -\overline{\gamma}\ \epsilon\ -\gamma\ \overline{\epsilon}\ -\kappa\ \nu\ +\beta\ \pi\ +\alpha\ \overline{\pi}\ +\alpha\ \tau\ +\pi\ \tau\ +\beta\ \overline{\tau}\ +\Delta\,\epsilon\ ,$

$D\,\lambda\ ==\ -\Phi_{20}\ -3\,\epsilon\ \lambda\ +\overline{\epsilon}\ \lambda\ -\overline{\kappa}\ \nu\ +\alpha\ \pi\ -\overline{\beta}\ \pi\ +\pi^{\,2}\ +\lambda\ \rho\ +\mu\ \overline{\sigma}\ +\overline{\delta}\,\pi\ ,$

$D\,\mu\ ==\ -2\,\Lambda\ +\Psi_2\ -\epsilon\ \mu\ -\overline{\epsilon}\ \mu\ -\kappa\ \nu\ -\overline{\alpha}\ \pi\ +\beta\ \pi\ +\pi\ \overline{\pi}\ +\mu\ \overline{\rho}\ +\lambda\ \sigma\ +\delta\,\pi\ ,$

$D\,\nu\ ==\ -\Phi_{21}\ +\Psi_3\ -3\,\epsilon\ \nu\ -\overline{\epsilon}\ \nu\ +\gamma\ \pi\ -\overline{\gamma}\ \pi\ +\mu\ \pi\ +\lambda\ \overline{\pi}\ +\lambda\ \tau\ +\mu\ \overline{\tau}\ +\Delta\,\pi\ ,$

$D\,\rho\ ==\ -\Phi_{00}\ -3\,\alpha\ \kappa\ -\overline{\beta}\ \kappa\ +\kappa\ \pi\ +\epsilon\ \rho\ +\overline{\epsilon}\ \rho\ +\rho^{\,2}\ +\sigma\ \overline{\sigma}\ -\overline{\kappa}\ \tau\ +\overline{\delta}\,\kappa\ ,$

$D\,\sigma\ ==\ \Psi_0\ -\overline{\alpha}\ \kappa\ -3\,\beta\ \kappa\ +\kappa\ \overline{\pi}\ +3\,\epsilon\ \sigma\ -\overline{\epsilon}\ \sigma\ +\rho\ \sigma\ +\overline{\rho}\ \sigma\ -\kappa\ \tau\ +\delta\,\kappa\ ,$

$D\,\tau\ ==\ -\Phi_{01}\ +\Psi_1\ -3\,\gamma\ \kappa\ -\overline{\gamma}\ \kappa\ +\overline{\pi}\ \rho\ +\pi\ \sigma\ +\epsilon\ \tau\ -\overline{\epsilon}\ \tau\ +\rho\ \tau\ +\sigma\ \overline{\tau}\ +\Delta\,\kappa\ ,$

$\triangle\,\alpha\ ==\ -\Psi_3\ +\overline{\beta}\ \gamma\ +\alpha\ \overline{\gamma}\ -\beta\ \lambda\ -\alpha\ \overline{\mu}\ +\epsilon\ \nu\ +\nu\ \rho\ -\lambda\ \tau\ -\gamma\ \overline{\tau}\ +\overline{\delta}\,\gamma\ ,$

$\triangle\,\beta\ ==\ \Phi_{12}\ +\overline{\alpha}\ \gamma\ +2\,\beta\ \gamma\ -\beta\ \overline{\gamma}\ -\alpha\ \overline{\lambda}\ -\beta\ \mu\ +\epsilon\ \overline{\nu}\ +\nu\ \sigma\ -\gamma\ \tau\ -\mu\ \tau\ +\delta\,\gamma\ ,$

$\triangle\,\lambda\ ==\ -\Psi_4\ -3\,\gamma\ \lambda\ +\overline{\gamma}\ \lambda\ -\lambda\ \mu\ -\lambda\ \overline{\mu}\ +3\,\alpha\ \nu\ +\overline{\beta}\ \nu\ +\nu\ \pi\ -\nu\ \overline{\tau}\ +\overline{\delta}\,\nu\ ,$

$\triangle\,\mu\ ==\ \Phi_{22}\ -\lambda\ \overline{\lambda}\ -\gamma\ \mu\ -\overline{\gamma}\ \mu\ -\mu^{\,2}\ +\overline{\alpha}\ \nu\ +3\,\beta\ \nu\ +\overline{\nu}\ \pi\ -\nu\ \tau\ +\delta\,\nu\ ,$

$\triangle\,\rho\ ==\ 2\,\Lambda\ -\Psi_2\ +\kappa\ \nu\ +\gamma\ \rho\ +\overline{\gamma}\ \rho\ -\overline{\mu}\ \rho\ -\lambda\ \sigma\ -\alpha\ \tau\ +\overline{\beta}\ \tau\ -\tau\ \overline{\tau}\ +\overline{\delta}\,\tau\ ,$

$\triangle\,\sigma\ ==\ \Phi_{02}\ +\kappa\ \overline{\nu}\ -\overline{\lambda}\ \rho\ +3\,\gamma\ \sigma\ -\overline{\gamma}\ \sigma\ -\mu\ \sigma\ +\overline{\alpha}\ \tau\ -\beta\ \tau\ -\tau^{\,2}\ +\delta\,\tau\ ,$

$\delta\,\alpha\ ==\ -\Lambda\ -\Phi_{11}\ -\Psi_2\ +\alpha\ \overline{\alpha}\ -2\,\alpha\ \beta\ +\beta\ \overline{\beta}\ +\epsilon\ \mu\ -\epsilon\ \overline{\mu}\ +\gamma\ \rho\ +\mu\ \rho\ -\gamma\ \overline{\rho}\ -\lambda\ \sigma\ +\overline{\delta}\,\beta\ ,$

$\delta\,\lambda\ ==\ -\Phi_{21}\ -\Psi_3\ +\overline{\alpha}\ \lambda\ -3\,\beta\ \lambda\ +\alpha\ \mu\ +\overline{\beta}\ \mu\ +\mu\ \pi\ -\overline{\mu}\ \pi\ +\nu\ \rho\ -\nu\ \overline{\rho}\ +\overline{\delta}\,\mu\ ,$

$\delta\,\rho\ ==\ -\Phi_{01}\ -\Psi_1\ +\kappa\ \mu\ -\kappa\ \overline{\mu}\ +\overline{\alpha}\ \rho\ +\beta\ \rho\ -3\,\alpha\ \sigma\ +\overline{\beta}\ \sigma\ +\rho\ \tau\ -\overline{\rho}\ \tau\ +\overline{\delta}\,\sigma\ \}\ \gg$
~/calculations/NP5D/NP4D

■ **Bianchi identities**

We take next the components of the second Bianchi identity in the null tetrad.

In[270]:= **3 * Antisymmetrize[CD[-a]@RiemannCD[-b, -c, -d, -f], {-a, -b, -c}] == 0**

Out[270]= $\dfrac{1}{2}\ (\nabla_a\,R[\nabla]_{bcdf}\ -\nabla_a\,R[\nabla]_{cbdf}\ -\nabla_b\,R[\nabla]_{acdf}\ +\nabla_b\,R[\nabla]_{cadf}\ +\nabla_c\,R[\nabla]_{abdf}\ -\nabla_c\,R[\nabla]_{badf})\ ==\ 0$

In[271]:= **ToCanonical /@ %**

Out[271]= $\nabla_a\,R[\nabla]_{bcdf}\ -\nabla_b\,R[\nabla]_{acdf}\ +\nabla_c\,R[\nabla]_{abdf}\ ==\ 0$

In[272]:= **ChangeCovD[♯, CD, PDNP] & /@ %**

Out[272]= $-\Gamma[\nabla,\mathcal{D}]^h{}_{cf}\,R[\nabla]_{abdh}\ -\Gamma[\nabla,\mathcal{D}]^h{}_{cd}\,R[\nabla]_{abhf}\ +\Gamma[\nabla,\mathcal{D}]^h{}_{bf}\,R[\nabla]_{acdh}\ +\Gamma[\nabla,\mathcal{D}]^h{}_{bd}\,R[\nabla]_{achf}\ +\Gamma[\nabla,\mathcal{D}]^h{}_{bc}\,R[\nabla]_{ahdf}\ -$
$\Gamma[\nabla,\mathcal{D}]^h{}_{cb}\,R[\nabla]_{ahdf}\ -\Gamma[\nabla,\mathcal{D}]^h{}_{af}\,R[\nabla]_{bcdh}\ -\Gamma[\nabla,\mathcal{D}]^h{}_{ad}\,R[\nabla]_{bchf}\ -\Gamma[\nabla,\mathcal{D}]^h{}_{ac}\,R[\nabla]_{bhdf}\ -\Gamma[\nabla,\mathcal{D}]^h{}_{ca}\,R[\nabla]_{hbdf}\ -$
$\Gamma[\nabla,\mathcal{D}]^h{}_{ab}\,R[\nabla]_{hcdf}\ +\Gamma[\nabla,\mathcal{D}]^h{}_{ba}\,R[\nabla]_{hcdf}\ +\mathcal{D}_a\,R[\nabla]_{bcdf}\ -\mathcal{D}_b\,R[\nabla]_{acdf}\ +\mathcal{D}_c\,R[\nabla]_{abdf}\ ==\ 0$

In[273]:= **ToCanonical /@ %**

Out[273]= $-\Gamma[\nabla,\mathcal{D}]^h{}_{cf}\,R[\nabla]_{abdh}\ +\Gamma[\nabla,\mathcal{D}]^h{}_{cd}\,R[\nabla]_{abfh}\ +\Gamma[\nabla,\mathcal{D}]^h{}_{bf}\,R[\nabla]_{acdh}\ -\Gamma[\nabla,\mathcal{D}]^h{}_{bd}\,R[\nabla]_{acfh}\ +\Gamma[\nabla,\mathcal{D}]^h{}_{bc}\,R[\nabla]_{ahdf}\ -$
$\Gamma[\nabla,\mathcal{D}]^h{}_{cb}\,R[\nabla]_{ahdf}\ -\Gamma[\nabla,\mathcal{D}]^h{}_{af}\,R[\nabla]_{bcdh}\ +\Gamma[\nabla,\mathcal{D}]^h{}_{ad}\,R[\nabla]_{bcfh}\ -\Gamma[\nabla,\mathcal{D}]^h{}_{ac}\,R[\nabla]_{bhdf}\ +\Gamma[\nabla,\mathcal{D}]^h{}_{ca}\,R[\nabla]_{bhdf}\ +$
$\Gamma[\nabla,\mathcal{D}]^h{}_{ab}\,R[\nabla]_{chdf}\ -\Gamma[\nabla,\mathcal{D}]^h{}_{ba}\,R[\nabla]_{chdf}\ +\mathcal{D}_a\,R[\nabla]_{bcdf}\ -\mathcal{D}_b\,R[\nabla]_{acdf}\ +\mathcal{D}_c\,R[\nabla]_{abdf}\ ==\ 0$

In[274]:= **% /. ChristoffelCDPDNP -> ChristoffelCDePDNP**

Out[274]= $-\Gamma[\nabla,\mathcal{D}]^h{}_{cf}\,R[\nabla]_{abdh}\ +\Gamma[\nabla,\mathcal{D}]^h{}_{cd}\,R[\nabla]_{abfh}\ +\Gamma[\nabla,\mathcal{D}]^h{}_{bf}\,R[\nabla]_{acdh}\ -\Gamma[\nabla,\mathcal{D}]^h{}_{bd}\,R[\nabla]_{acfh}\ +\Gamma[\nabla,\mathcal{D}]^h{}_{bc}\,R[\nabla]_{ahdf}\ -$
$\Gamma[\nabla,\mathcal{D}]^h{}_{cb}\,R[\nabla]_{ahdf}\ -\Gamma[\nabla,\mathcal{D}]^h{}_{af}\,R[\nabla]_{bcdh}\ +\Gamma[\nabla,\mathcal{D}]^h{}_{ad}\,R[\nabla]_{bcfh}\ -\Gamma[\nabla,\mathcal{D}]^h{}_{ac}\,R[\nabla]_{bhdf}\ +\Gamma[\nabla,\mathcal{D}]^h{}_{ca}\,R[\nabla]_{bhdf}\ +$
$\Gamma[\nabla,\mathcal{D}]^h{}_{ab}\,R[\nabla]_{chdf}\ -\Gamma[\nabla,\mathcal{D}]^h{}_{ba}\,R[\nabla]_{chdf}\ +\mathcal{D}_a\,R[\nabla]_{bcdf}\ -\mathcal{D}_b\,R[\nabla]_{acdf}\ +\mathcal{D}_c\,R[\nabla]_{abdf}\ ==\ 0$

In[275]:= **ToBasis[NP] /@ ToBasis[NP] /@ %**

Out[275]= $-\Gamma[\nabla,\mathcal{D}]^{h}{}_{cf} R[\nabla]_{abdh} + \Gamma[\nabla,\mathcal{D}]^{h}{}_{cd} R[\nabla]_{abfh} + \Gamma[\nabla,\mathcal{D}]^{h}{}_{bf} R[\nabla]_{acdh} -$
$\Gamma[\nabla,\mathcal{D}]^{h}{}_{bd} R[\nabla]_{acfh} + \Gamma[\nabla,\mathcal{D}]^{h}{}_{bc} R[\nabla]_{ahdf} - \Gamma[\nabla,\mathcal{D}]^{h}{}_{cb} R[\nabla]_{ahdf} - \Gamma[\nabla,\mathcal{D}]^{h}{}_{af} R[\nabla]_{bcdh} +$
$\Gamma[\nabla,\mathcal{D}]^{h}{}_{ad} R[\nabla]_{bcfh} - \Gamma[\nabla,\mathcal{D}]^{h}{}_{ac} R[\nabla]_{bhdf} + \Gamma[\nabla,\mathcal{D}]^{h}{}_{ca} R[\nabla]_{bhdf} + \Gamma[\nabla,\mathcal{D}]^{h}{}_{ab} R[\nabla]_{chdf} -$
$\Gamma[\nabla,\mathcal{D}]^{h}{}_{ba} R[\nabla]_{chdf} + \mathcal{D}_{a} R[\nabla]_{bcdf} - \mathcal{D}_{b} R[\nabla]_{acdf} + \mathcal{D}_{c} R[\nabla]_{abdf} == 0$

In[276]:= **TraceBasisDummy /@ %;**

In[277]:= **Flatten@ComponentArray@%;**

In[278]:= **NPBianchi = Union[Simplify //@ %];**

Not all the equations here are independent.

In[279]:= **Dagger /@ Flatten@Outer[PDNP[#1]@#2 &,**
{{1, -NP}, {2, -NP}, {3, -NP}, {4, -NP}}, NPCVComponents[[Range[1, 5]]], 1]

Out[279]= $\left\{ D\,\overline{\Psi}_0 \,,\, D\,\overline{\Psi}_1 \,,\, D\,\overline{\Psi}_2 \,,\, D\,\overline{\Psi}_3 \,,\, D\,\overline{\Psi}_4 \,,\, \triangle\,\overline{\Psi}_0 \,,\, \triangle\,\overline{\Psi}_1 \,,\, \triangle\,\overline{\Psi}_2 \,,\, \triangle\,\overline{\Psi}_3 \,,\right.$
$\left. \triangle\,\overline{\Psi}_4 \,,\, \overline{\delta}\,\overline{\Psi}_0 \,,\, \overline{\delta}\,\overline{\Psi}_1 \,,\, \overline{\delta}\,\overline{\Psi}_2 \,,\, \overline{\delta}\,\overline{\Psi}_3 \,,\, \overline{\delta}\,\overline{\Psi}_4 \,,\, \delta\,\overline{\Psi}_0 \,,\, \delta\,\overline{\Psi}_1 \,,\, \delta\,\overline{\Psi}_2 \,,\, \delta\,\overline{\Psi}_3 \,,\, \delta\,\overline{\Psi}_4 \right\}$

In[280]:= **Join[%, {PDNP[{4, -NP}]@LambdaCDe[]}]**

Out[280]= $\left\{ D\,\overline{\Psi}_0 \,,\, D\,\overline{\Psi}_1 \,,\, D\,\overline{\Psi}_2 \,,\, D\,\overline{\Psi}_3 \,,\, D\,\overline{\Psi}_4 \,,\, \triangle\,\overline{\Psi}_0 \,,\, \triangle\,\overline{\Psi}_1 \,,\, \triangle\,\overline{\Psi}_2 \,,\, \triangle\,\overline{\Psi}_3 \,,\, \triangle\,\overline{\Psi}_4 \,,\right.$
$\left. \overline{\delta}\,\overline{\Psi}_0 \,,\, \overline{\delta}\,\overline{\Psi}_1 \,,\, \overline{\delta}\,\overline{\Psi}_2 \,,\, \overline{\delta}\,\overline{\Psi}_3 \,,\, \overline{\delta}\,\overline{\Psi}_4 \,,\, \delta\,\overline{\Psi}_0 \,,\, \delta\,\overline{\Psi}_1 \,,\, \delta\,\overline{\Psi}_2 \,,\, \delta\,\overline{\Psi}_3 \,,\, \delta\,\overline{\Psi}_4 \,,\, \overline{\delta}\,\Lambda \right\}$

In[281]:= **List @@ Eliminate[NPBianchi, %];**

The final result for the Bianchi identities is

In[282]:= **NPBianchi = Map[FullSimplify, %, {2}]**

Out[282]= $\Big\{ D \Phi_{01} \; == \; 2 \Psi_1 \; \epsilon \; - 2 \Phi_{11} \; \kappa \; - 3 \Psi_2 \; \kappa \; - \Phi_{02} \; \overline{\kappa} \; + \Psi_0 \; (-4 \alpha \; + \pi \;) +$

$\Phi_{00} \;\; (-2 \; (\overline{\alpha} \; + \beta \;) + \overline{\pi} \;) + 4 \Psi_1 \; \rho \; + 2 \Phi_{01} \;\; (\epsilon \; + \overline{\rho} \;) + 2 \Phi_{10} \;\; \sigma \; - D \Psi_1 \; + \delta \Phi_{00} \; + \overline{\delta} \Psi_0 \;,$

$D \Phi_{02} \; == \; - 2 \Phi_{01} \; \beta \; + 2 \Psi_1 \; \beta \; - 2 \Phi_{12} \; \kappa \; - \Phi_{00} \; \overline{\lambda} \; + \Psi_0 \;\; (-4 \gamma \; + \mu \;) + 2 \Phi_{01} \;\; \overline{\pi} \; +$

$\Phi_{02} \;\; (2 \epsilon \; - 2 \overline{\epsilon} \; + \overline{\rho} \;) + 2 \Phi_{11} \;\; \sigma \; - 3 \Psi_2 \; \sigma \; + 4 \Psi_1 \; \tau \; + \triangle \Psi_0 \; + \delta \Phi_{01} \; - \delta \Psi_1 \;,$

$D \Phi_{11} \; == \; - (\Phi_{21} \; + 2 \Psi_3 \;) \; \kappa \; - \Phi_{12} \; \overline{\kappa} \; - \Psi_0 \; \lambda \; - \Phi_{00} \; \mu \; + \Phi_{01} \; \pi \; + 2 \Psi_1 \;\; (-\alpha \; + \pi \;) +$

$\Phi_{10} \;\; (-2 \overline{\alpha} \; + \overline{\pi} \;) + 3 \Psi_2 \; \rho \; + 2 \Phi_{11} \;\; \overline{\rho} \; + \Phi_{20} \;\; \sigma \; - D \Lambda \; - D \Psi_2 \; + \delta \Phi_{10} \; + \overline{\delta} \Psi_1 \;,$

$D \Phi_{22} \; == \; - \Phi_{20} \;\; \overline{\lambda} \; - 2 \Phi_{11} \; \mu \; + 3 \Psi_2 \; \mu \; - 2 \Psi_1 \; \nu \; + 2 \Phi_{12} \; \pi \; + 2 \Phi_{21} \;\; (\beta \; + \overline{\pi} \;) +$

$\Phi_{22} \;\; (-2 \;(\epsilon \; + \overline{\epsilon} \;) + \overline{\rho} \;) - \Psi_4 \; \sigma \; + 2 \Psi_3 \;\; (-\beta \; + \tau \;) - 2 \; (\triangle \Lambda \;) + \triangle \Psi_2 \; + \delta \Phi_{21} \; - \delta \Psi_3 \;,$

$D \Psi_4 \; == \; - 2 \Phi_{21} \; \alpha \; + 2 \Psi_3 \; \alpha \; - 4 \Psi_4 \; \epsilon \; + 2 \Phi_{11} \; \lambda \; - 3 \Psi_2 \; \lambda \; + \Phi_{20} \;\; (2 \gamma \; - 2 \overline{\gamma} \; + \overline{\mu} \;) -$

$2 \Phi_{10} \;\; \nu \; + 4 \Psi_3 \; \pi \; + \Psi_4 \; \rho \; - \Phi_{22} \;\; \overline{\sigma} \; + 2 \Phi_{21} \;\; \overline{\tau} \; + \triangle \Phi_{20} \; - \overline{\delta} \Phi_{21} \; + \overline{\delta} \Psi_3 \;,$

$\triangle \Phi_{00} \; == \; 2 \Phi_{00} \;\; (\gamma \; + \overline{\gamma} \;) + 2 \Psi_3 \; \kappa \; + \Psi_0 \; \lambda \; - \Phi_{00} \; \overline{\mu} \; + 2 \Psi_1 \;\; (\alpha \; - \pi \;) + 2 \Phi_{11} \;\; \rho \; -$

$3 \Psi_2 \; \rho \; + \Phi_{02} \;\; \overline{\sigma} \; - 2 \Phi_{10} \;\; \tau \; - 2 \Phi_{01} \;\; (\alpha \; + \overline{\tau} \;) - 2 \; (D \Lambda \;) + D \Psi_2 \; + \overline{\delta} \Phi_{01} \; - \overline{\delta} \Psi_1 \;,$

$\triangle \Phi_{11} \; == \; 2 \Psi_3 \; \beta \; + 2 \Phi_{12} \;\; \overline{\beta} \; - \Phi_{02} \; \lambda \; - 3 \Psi_2 \; \mu \; - 2 \Phi_{11} \;\; \overline{\mu} \; + \Phi_{01} \; \nu \; + 2 \Psi_1 \; \nu \; + \Phi_{10} \;\; \overline{\nu} \; +$

$\Phi_{22} \;\; \rho \; + \Psi_4 \; \sigma \; - \Phi_{21} \;\; \tau \; - 2 \Psi_3 \; \tau \; - \Phi_{12} \;\; \overline{\tau} \; - \triangle \Lambda \; - \triangle \Psi_2 \; + \delta \Psi_3 \; + \overline{\delta} \Phi_{12} \;,$

$\triangle \Phi_{21} \; == \; 2 \Phi_{22} \;\; \left(\alpha \; + \overline{\beta} \;\right) - 2 \Phi_{12} \;\; \lambda \; - 2 \; (\Psi_3 \;\; (\gamma \; + 2 \mu \;) + \Phi_{21} \;\; (\gamma \; + \overline{\mu} \;)) + 2 \Phi_{11} \;\; \nu \; +$

$3 \Psi_2 \; \nu \; + \Phi_{20} \;\; \overline{\nu} \; + \Psi_4 \;\; (4 \beta \; - \tau \;) - \Phi_{22} \;\; \overline{\tau} \; - \triangle \Psi_3 \; + \delta \Psi_4 \; + \overline{\delta} \Phi_{22} \;,$

$\delta \Phi_{11} \; == \; 2 \Psi_1 \; \gamma \; + 2 \Phi_{12} \;\; \overline{\epsilon} \; + \Phi_{22} \;\; \kappa \; + \Phi_{10} \;\; \overline{\lambda} \; + \Phi_{01} \; \mu \; - 2 \Psi_1 \; \mu \; + \Psi_0 \; \nu \; - \Phi_{02} \; \pi \; -$

$2 \Phi_{11} \;\; \overline{\pi} \; - \Phi_{12} \;\; \overline{\rho} \; - \Phi_{21} \;\; \sigma \; + 2 \Psi_3 \; \sigma \; - 3 \Psi_2 \; \tau \; + D \Phi_{12} \; - \triangle \Psi_1 \; + \delta \Lambda \; + \delta \Psi_2 \;,$

$\overline{\delta} \Phi_{02} \; == \; - 2 \Phi_{01} \; \gamma \; - 2 \Psi_1 \; \gamma \; + 2 \Psi_1 \; \mu \; + 2 \Phi_{01} \;\; \overline{\mu} \; - \Psi_0 \; \nu \; - \Phi_{00} \;\; \overline{\nu} \; - 2 \Phi_{12} \;\; \rho \; - 2 \Psi_3 \; \sigma \; +$

$2 \Phi_{11} \;\; \tau \; + 3 \Psi_2 \; \tau \; + \Phi_{02} \;\; \left(2 \alpha \; - 2 \overline{\beta} \; + \overline{\tau} \;\right) + \triangle \Phi_{01} \; + \triangle \Psi_1 \; + 2 \; (\delta \Lambda \;) - \delta \Psi_2 \;,$

$2 \;\left(\overline{\delta} \Phi_{11} \;\right) == \; - 6 \Psi_3 \; \epsilon \; - 3 \Psi_4 \; \kappa \; - \Phi_{22} \;\; \overline{\kappa} \; + 2 \Phi_{01} \; \lambda \; - 6 \Psi_1 \; \lambda \; - 2 \Phi_{10} \;\; (2 \overline{\gamma} \; + \mu \; - \overline{\mu} \;) -$

$2 \Phi_{00} \;\; \nu \; + 2 \Phi_{11} \;\; \pi \; + 9 \Psi_2 \; \pi \; + 6 \Psi_3 \; \rho \; - 2 \Phi_{21} \;\; (\epsilon \; + \rho \; - \overline{\rho} \;) - 2 \Phi_{12} \;\; \overline{\sigma} \; +$

$\Phi_{20} \;\; (-2 \overline{\alpha} \; + 2 \beta \; + \overline{\pi} \; + 2 \tau \;) + 4 \Phi_{11} \;\; \overline{\tau} \; - D \Phi_{21} \; - 3 \; (D \Psi_3 \;) + 2 \; (\triangle \Phi_{10} \;) + \delta \Phi_{20} \; + 3 \;\left(\overline{\delta} \Psi_2 \;\right) \Big\}$

■ 3. The GHP formalism

The Geroch-Held-Penrose formalism (known as GHP formalism) is a particularisation of the Newman-Penrose formalism in which the spin coefficients are separated according to their *boost* and *spin weight*. Also the direcctional derivatives associated to the null tetrad are replaced by differential operators which behave well under boost and spin transformations and therefore we can speak of the boost and spin weight of these operators. This enables us to re-write the basic equations used in the Newman-Penrose formalism in terms of boost-weighted quantities only. In this section we show how this is accomplished in the framework of Spinors` and xCoba` .

■ Boost and spin weights

The GHP formalism only admits *weighted quantities*. In this subsection we introduce a command, called **GHPWeightOf**, which computes the boost and spin weight of any quantity. These are integers which are kept in a list. The command **GHPWeightOf** also checks that the GHP weight of the terms of any sum all agree, throwing an error otherwise.

In[283]:= **GHPWeightOf::usage =**
    "GHPWeightOf[expr] returns the GHP boost and spin weight of any expression.
      If the expression is not weighted, then an error message is returned";

Weights of integers, rationals and complex numbers.

In[284]:= **GHPWeightOf[num_?IntegerQ] := {0, 0};**
    GHPWeightOf[_Rational] := {0, 0};
    GHPWeightOf[_Complex] := {0, 0};

Weights of products.

In[287]:= `GHPWeightOf[expr_Times] := Apply[Plus, GHPWeightOf /@ Level[expr, {1}]];`

Weights of sums

In[288]:= `GHPWeightOf@expr_Plus := If[Length@Union[GHPWeightOf /@ Level[expr, 1]] > 1,`
`    Throw@Message[GHPWeightOf::Error], First@Union[GHPWeightOf /@ Level[expr, 1]]];`
`GHPWeightOf::Error = "Inconsistent weights on Plus expression";`

Weights of powers

In[290]:= `GHPWeightOf[expr_ ^ (n_ ? IntegerQ)] := n GHPWeightOf@expr;`

- **Weights of the spin coefficients**

Some of the spin coefficients are weighted quantities. These are in the following list.

In[291]:= `GHPSpinCoefficients = NPSpinCoefficients[[#]] & /@ Range[5, 12]`

Out[291]= $\{\kappa, \lambda, \mu, \nu, \pi, \rho, \sigma, \tau\}$

These are the spin and boost weights of these quantities :

In[292]:= `{GHPWeightOf@κ Dyad[] ^= {3, 1}, GHPWeightOf@λ Dyad[] ^= {-3, 1},`
`    GHPWeightOf@μ Dyad[] ^= {-1, -1}, GHPWeightOf@ν Dyad[] ^= {-3, -1},`
`    GHPWeightOf@π Dyad[] ^= {-1, 1}, GHPWeightOf@ρ Dyad[] ^= {1, 1},`
`    GHPWeightOf@σ Dyad[] ^= {3, -1}, GHPWeightOf@τ Dyad[] ^= {1, -1}};`

From here we compute the weights of the complex conjugate of these quantities

In[293]:= `Dagger /@ GHPSpinCoefficients`

Out[293]= $\{\overline{\kappa}, \overline{\lambda}, \overline{\mu}, \overline{\nu}, \overline{\pi}, \overline{\rho}, \overline{\sigma}, \overline{\tau}\}$

In[294]:= `UpSet[GHPWeightOf@#, Reverse@GHPWeightOf@Dagger@#] & /@ %`

Out[294]= `{{1, 3}, {1, -3}, {-1, -1}, {-1, -3}, {1, -1}, {1, 1}, {-1, 3}, {-1, 1}}`

- **Weights of the components of the curvature spinors**

It is easy to deduce the weights of the components of the curvature spinors. These are their values.

In[295]:= `{GHPWeightOf@PsiCDe0Dyad[] ^= {4, 0},`
`    GHPWeightOf@PsiCDe1Dyad[] ^= {2, 0}, GHPWeightOf@PsiCDe2Dyad[] ^= {0, 0},`
`    GHPWeightOf@PsiCDe3Dyad[] ^= {-2, 0}, GHPWeightOf@PsiCDe4Dyad[] ^= {-4, 0}};`

In[296]:= `{GHPWeightOf@PsiCDe0Dyad†[] ^= Reverse@{4, 0}, GHPWeightOf@PsiCDe1Dyad†[] ^= Reverse@{2, 0},`
`    GHPWeightOf@PsiCDe2Dyad†[] ^= Reverse@{0, 0}, GHPWeightOf@PsiCDe3Dyad†[] ^=`
`     Reverse@{-2, 0}, GHPWeightOf@PsiCDe4Dyad†[] ^= Reverse@{-4, 0}};`

In[297]:= `{GHPWeightOf@PhiCDe00Dyad[] ^= {2, 2},`
`    GHPWeightOf@PhiCDe10Dyad[] ^= {0, 2}, GHPWeightOf@PhiCDe01Dyad[] ^= {2, 0},`
`    GHPWeightOf@PhiCDe11Dyad[] ^= {0, 0}, GHPWeightOf@PhiCDe20Dyad[] ^= {-2, 2},`
`    GHPWeightOf@PhiCDe21Dyad[] ^= {-2, 0}, GHPWeightOf@PhiCDe02Dyad[] ^= {2, -2},`
`    GHPWeightOf@PhiCDe12Dyad[] ^= {0, -2}, GHPWeightOf@PhiCDe22Dyad[] ^= {-2, -2}};`

In[298]:= `GHPWeightOf@LambdaCDe[] ^= {0, 0};`

GHP differential operators

The GHP differential operators are constructed by adding some extra terms to the four NP direccional derivatives. The result is a set of four differential operators having good transformation properties under spin and boost.

The GHP differential operators can be regarded as the "components" of a certain differential operator which we define next. This operator is not a true covariant derivative but just a differential operator acting on scalars, so we need to use some primitive xAct` instructions in order to create it. We use the notation Θ_a as a representation for it.

```
In[299]:= CovDQ[Θ] ^= True;
       xAct`xTensor`Private`MakeLinearDerivative[{Θ[A_], Θ[A]}, True];
       SymmetryGroupOfCovD[Θ] ^= StrongGenSet[{}, GenSet[]];
       ManifoldOfCovD[Θ] ^= ManifoldOfCovD[CDe];
       VBundlesOfCovD[Θ] ^= {};
       Dagger[Θ[A_][expr_]] ^:= Θ[DaggerIndex[A]][Dagger[expr]];
       SymbolOfCovD[Θ] ^= {"Θ"};
```

Formatting of the GHP operators with their standard symbols.

```
In[306]:= xTensorFormStop[CovD]
       FormatBasis[Θ[{1, -NP}], "þ"];
       FormatBasis[Θ[{2, -NP}], "þ'"];
       FormatBasis[Θ[{3, -NP}], "ð"];
       FormatBasis[Θ[{4, -NP}], "ð'"];
       xTensorFormStart[CovD]
```

The following constants represent respectively generic boost and spin weights.

```
In[312]:= DefConstantSymbol[w]
```

** DefConstantSymbol: Defining constant symbol w.

```
In[313]:= DefConstantSymbol[z]
```

** DefConstantSymbol: Defining constant symbol z.

We introduce a complex scalar of weight (p,q).

```
In[314]:= DefSpinor[η[LI@{p, q}], M4]
```

** DefTensor: Defining tensor η[LI[{p, q}]].

** DefTensor: Defining tensor η †[LI[{p, q}]].

We specify the weight of the scalar quantity $\eta^{\{w,\ z\}}$ (note that the integers w and z are zero weighted).

```
In[315]:= GHPWeightOf@η[LI@{w, z}] ^= {w, z};
       GHPWeightOf@w ^= {0, 0}; GHPWeightOf@z ^= {0, 0};
```

The action of the differential operator Θ_a on a weighted scalar $\eta^{\{w, z\}}$ is, by definition

In[317]:= **{Θ[{1, -NP}][η[LI[{w, z}]]] ->**
(-(w * ϵ Dyad†[]) - z * ϵ Dyad†[]) * η[LI[{w, z}]] + PDNP[{1, -NP}][η[LI[{w, z}]]],
Θ[{2, -NP}][η[LI[{w, z}]]] -> (-(w * γ Dyad†[]) - z * γ Dyad†[]) * η[LI[{w, z}]] +
PDNP[{2, -NP}][η[LI[{w, z}]]],
Θ[{3, -NP}][η[LI[{w, z}]]] -> (-(z * α Dyad†[]) - w * β Dyad†[]) * η[LI[{w, z}]] +
PDNP[{3, -NP}][η[LI[{w, z}]]],
Θ[{4, -NP}][η[LI[{w, z}]]] -> (-(w * α Dyad†[]) - z * β Dyad†[]) * η[LI[{w, z}]] +
PDNP[{4, -NP}][η[LI[{w, z}]]]}

Out[317]= $\left\{ þ\, \eta^{\{w, z\}} \to (-w\, \epsilon\ -z\, \overline{\epsilon}\ )\, \eta^{\{w, z\}} + D\, \eta^{\{w, z\}},\ þ'\, \eta^{\{w, z\}} \to (-w\, \gamma\ -z\, \overline{\gamma}\ )\, \eta^{\{w, z\}} + \triangle\, \eta^{\{w, z\}}, \right.$
$\left. ð\, \eta^{\{w, z\}} \to (-z\, \overline{\alpha}\ -w\, \beta\ )\, \eta^{\{w, z\}} + \delta\, \eta^{\{w, z\}},\ ð'\, \eta^{\{w, z\}} \to \left(-w\, \alpha\ -z\, \overline{\beta}\ \right)\, \eta^{\{w, z\}} + \overline{\delta}\, \eta^{\{w, z\}} \right\}$

In[318]:= **GHPDerivativesToFrame = %;**

In[319]:= **% /. Rule → Equal;**

In[320]:= **Solve[%, Cases[%, PDNP[_]@_, Infinity]] // Flatten**

Out[320]= $\left\{ D\, \eta^{\{w, z\}} \to w\, \epsilon\ \ \eta^{\{w, z\}} + z\, \overline{\epsilon}\ \ \eta^{\{w, z\}} + þ\, \eta^{\{w, z\}},\ \triangle\, \eta^{\{w, z\}} \to w\, \gamma\ \ \eta^{\{w, z\}} + z\, \overline{\gamma}\ \ \eta^{\{w, z\}} + þ'\, \eta^{\{w, z\}}, \right.$
$\left. \delta\, \eta^{\{w, z\}} \to z\, \overline{\alpha}\ \ \eta^{\{w, z\}} + w\, \beta\ \ \eta^{\{w, z\}} + ð\, \eta^{\{w, z\}},\ \overline{\delta}\, \eta^{\{w, z\}} \to w\, \alpha\ \ \eta^{\{w, z\}} + z\, \overline{\beta}\ \ \eta^{\{w, z\}} + ð'\, \eta^{\{w, z\}} \right\}$

In[321]:= **FrameToGHPDerivatives = %;**

We transform previous rules in delayed rules with patterns which can be used for any weighted quantity

In[322]:= **GHPDerivativesToFrame /. Rule -> RuleDelayed /. η[LI@{w, z}] -> η**

Out[322]= $\left\{ þ\, \eta :\to (-w\, \epsilon\ -z\, \overline{\epsilon}\ )\, \eta + D\, \eta,\ þ'\, \eta :\to (-w\, \gamma\ -z\, \overline{\gamma}\ )\, \eta + \triangle\, \eta, \right.$
$\left. ð\, \eta :\to (-z\, \overline{\alpha}\ -w\, \beta\ )\, \eta + \delta\, \eta,\ ð'\, \eta :\to \left(-w\, \alpha\ -z\, \overline{\beta}\ \right)\, \eta + \overline{\delta}\, \eta \right\}$

In[323]:= **% /. Θ[comp__]@η :> Θ[comp]@η _**

Out[323]= $\left\{ þ\, \eta_{\_} :\to (-w\, \epsilon\ -z\, \overline{\epsilon}\ )\, \eta + D\, \eta,\ þ'\, \eta_{\_} :\to (-w\, \gamma\ -z\, \overline{\gamma}\ )\, \eta + \triangle\, \eta, \right.$
$\left. ð\, \eta_{\_} :\to (-z\, \overline{\alpha}\ -w\, \beta\ )\, \eta + \delta\, \eta,\ ð'\, \eta_{\_} :\to \left(-w\, \alpha\ -z\, \overline{\beta}\ \right)\, \eta + \overline{\delta}\, \eta \right\}$

In[324]:= **GHPToNP = (% /. w :> First@GHPWeightOf@η /. z :> Last@GHPWeightOf@η)**

Out[324]= $\left\{ þ\, \eta_{\_} :\to (-\text{First}[\text{GHPWeightOf}[\eta]]\, \epsilon\ -\text{Last}[\text{GHPWeightOf}[\eta]]\, \overline{\epsilon}\ )\, \eta + D\, \eta, \right.$
$þ'\, \eta_{\_} :\to (-\text{First}[\text{GHPWeightOf}[\eta]]\, \gamma\ -\text{Last}[\text{GHPWeightOf}[\eta]]\, \overline{\gamma}\ )\, \eta + \triangle\, \eta,$
$ð\, \eta_{\_} :\to (-\text{Last}[\text{GHPWeightOf}[\eta]]\, \overline{\alpha}\ -\text{First}[\text{GHPWeightOf}[\eta]]\, \beta\ )\, \eta + \delta\, \eta,$
$\left. ð'\, \eta_{\_} :\to \left(-\text{First}[\text{GHPWeightOf}[\eta]]\, \alpha\ -\text{Last}[\text{GHPWeightOf}[\eta]]\, \overline{\beta}\ \right)\, \eta + \overline{\delta}\, \eta \right\}$

In[325]:= **FrameToGHPDerivatives /. Rule -> RuleDelayed /. η[LI@{w, z}] -> η**

Out[325]= $\left\{ D\, \eta :\to w\, \epsilon\ \ \eta + z\, \overline{\epsilon}\ \ \eta + þ\, \eta,\ \triangle\, \eta :\to w\, \gamma\ \ \eta + z\, \overline{\gamma}\ \ \eta + þ'\, \eta,\ \delta\, \eta :\to z\, \overline{\alpha}\ \ \eta + w\, \beta\ \ \eta + ð\, \eta,\ \overline{\delta}\, \eta :\to w\, \alpha\ \ \eta + z\, \overline{\beta}\ \ \eta + ð'\, \eta \right\}$

In[326]:= **% /. PDNP[comp__]@η :> PDNP[comp]@η _**

Out[326]= $\left\{ D\, \eta_{\_} :\to w\, \epsilon\ \ \eta + z\, \overline{\epsilon}\ \ \eta + þ\, \eta,\ \triangle\, \eta_{\_} :\to w\, \gamma\ \ \eta + z\, \overline{\gamma}\ \ \eta + þ'\, \eta, \right.$
$\left. \delta\, \eta_{\_} :\to z\, \overline{\alpha}\ \ \eta + w\, \beta\ \ \eta + ð\, \eta,\ \overline{\delta}\, \eta_{\_} :\to w\, \alpha\ \ \eta + z\, \overline{\beta}\ \ \eta + ð'\, \eta \right\}$

In[327]:= **NPToGHP = (% /. w :> First@GHPWeightOf@η /. z :> Last@GHPWeightOf@η)**

Out[327]= $\Big\{ D\, \eta\, \_:\to \text{First}[\text{GHPWeightOf}[\eta]]\, \in\, \eta + \text{Last}[\text{GHPWeightOf}[\eta]]\, \overline{\in}\, \eta + \text{þ}\, \eta,$

$\triangle\, \eta\, \_:\to \text{First}[\text{GHPWeightOf}[\eta]]\, \gamma\, \eta + \text{Last}[\text{GHPWeightOf}[\eta]]\, \overline{\gamma}\, \eta + \text{þ'}\, \eta,$

$\delta\, \eta\, \_:\to \text{Last}[\text{GHPWeightOf}[\eta]]\, \overline{\alpha}\, \eta + \text{First}[\text{GHPWeightOf}[\eta]]\, \beta\, \eta + \text{ð}\, \eta,$

$\overline{\delta}\, \eta\, \_:\to \text{First}[\text{GHPWeightOf}[\eta]]\, \alpha\, \eta + \text{Last}[\text{GHPWeightOf}[\eta]]\, \overline{\beta}\, \eta + \text{ð'}\, \eta \Big\}$

Each of the quantities $\text{þ}\, \eta^{\{w, z\}}$, $\text{þ'}\, \eta^{\{w, z\}}$, $\text{ð}\, \eta^{\{w, z\}}$, $\text{ð'}\, \eta^{\{w, z\}}$ is a spin weighted quantity. These are the specific values for the weights of each of them.

In[328]:= **GHPWeightOf@Θ[{1, -NP}] ^= {1, 1};**
GHPWeightOf@Θ[{2, -NP}] ^= {-1, -1};
GHPWeightOf@Θ[{3, -NP}] ^= {1, -1};
GHPWeightOf@Θ[{4, -NP}] ^= {-1, 1};

We also need to teach GHPWeightOf how to behave when acting on differential expressions.

In[332]:= **GHPWeightOf[Θ[ind_ ? CIndexQ]@sc_] := {Part[GHPWeightOf@Θ[ind], 1] + Part[GHPWeightOf@sc, 1],**
Part[GHPWeightOf@Θ[ind], 2] + Part[GHPWeightOf@sc, 2]};

■ **Commutation of the GHP differential operators**

In this subsection we work out the commutation relations of each of the GHP operators. The commutator of two GHP operators is a new differential operator which acts on a generic spin weighted scalar $\eta^{\{w, z\}}$.

This is the commutation between two generic GHP operators.

In[333]:= **Θ[-{a, NP}]@Θ[-{b, NP}]@η[LI@{w, z}] - Θ[-{b, NP}]@Θ[-{a, NP}]@η[LI@{w, z}]**

Out[333]= $\Theta_a\, \Theta_b\, \eta^{\{w, z\}} - \Theta_b\, \Theta_a\, \eta^{\{w, z\}}$

We expand this into components in order to obtain the set of commutation relations.

In[334]:= **Delete[Union@Flatten@ComponentArray@%, 1]**

Out[334]= $\Big\{ \text{þ}\,\text{þ'}\, \eta^{\{w, z\}} - \text{þ'}\,\text{þ}\, \eta^{\{w, z\}},\ -\big(\text{þ}\,\text{þ'}\, \eta^{\{w, z\}}\big) + \text{þ'}\,\text{þ}\, \eta^{\{w, z\}},\ \text{þ}\,\text{ð}\, \eta^{\{w, z\}} - \text{ð}\,\text{þ}\, \eta^{\{w, z\}},$

$-\big(\text{þ}\,\text{ð}\, \eta^{\{w, z\}}\big) + \text{ð}\,\text{þ}\, \eta^{\{w, z\}},\ \text{þ'}\,\text{ð}\, \eta^{\{w, z\}} - \text{ð}\,\text{þ'}\, \eta^{\{w, z\}},\ -\big(\text{þ'}\,\text{ð}\, \eta^{\{w, z\}}\big) + \text{ð}\,\text{þ'}\, \eta^{\{w, z\}},$

$\text{þ}\,\text{ð'}\, \eta^{\{w, z\}} - \text{ð'}\,\text{þ}\, \eta^{\{w, z\}},\ -\big(\text{þ}\,\text{ð'}\, \eta^{\{w, z\}}\big) + \text{ð'}\,\text{þ}\, \eta^{\{w, z\}},\ \text{þ'}\,\text{ð'}\, \eta^{\{w, z\}} - \text{ð'}\,\text{þ'}\, \eta^{\{w, z\}},$

$-\big(\text{þ'}\,\text{ð'}\, \eta^{\{w, z\}}\big) + \text{ð'}\,\text{þ'}\, \eta^{\{w, z\}},\ \text{ð}\,\text{ð'}\, \eta^{\{w, z\}} - \text{ð'}\,\text{ð}\, \eta^{\{w, z\}},\ -\big(\text{ð}\,\text{ð'}\, \eta^{\{w, z\}}\big) + \text{ð'}\,\text{ð}\, \eta^{\{w, z\}} \Big\}$

In[335]:= **Union[%, SameTest → (-#1 === #2 &)]**

Out[335]= $\Big\{ \text{þ}\,\text{þ'}\, \eta^{\{w, z\}} - \text{þ'}\,\text{þ}\, \eta^{\{w, z\}},\ \text{þ}\,\text{ð}\, \eta^{\{w, z\}} - \text{ð}\,\text{þ}\, \eta^{\{w, z\}},\ \text{þ'}\,\text{ð}\, \eta^{\{w, z\}} - \text{ð}\,\text{þ'}\, \eta^{\{w, z\}},$

$\text{þ}\,\text{ð'}\, \eta^{\{w, z\}} - \text{ð'}\,\text{þ}\, \eta^{\{w, z\}},\ \text{þ'}\,\text{ð'}\, \eta^{\{w, z\}} - \text{ð'}\,\text{þ'}\, \eta^{\{w, z\}},\ \text{ð}\,\text{ð'}\, \eta^{\{w, z\}} - \text{ð'}\,\text{ð}\, \eta^{\{w, z\}} \Big\}$

In[336]:= **Union[%, SameTest → (Dagger@#1 === #2 &)]**

Out[336]= $\Big\{ \text{þ}\,\text{þ'}\, \eta^{\{w, z\}} - \text{þ'}\,\text{þ}\, \eta^{\{w, z\}},\ \text{þ}\,\text{ð}\, \eta^{\{w, z\}} - \text{ð}\,\text{þ}\, \eta^{\{w, z\}},\ \text{þ'}\,\text{ð}\, \eta^{\{w, z\}} - \text{ð}\,\text{þ'}\, \eta^{\{w, z\}},$

$\text{þ}\,\text{ð'}\, \eta^{\{w, z\}} - \text{ð'}\,\text{þ}\, \eta^{\{w, z\}},\ \text{þ'}\,\text{ð'}\, \eta^{\{w, z\}} - \text{ð'}\,\text{þ'}\, \eta^{\{w, z\}},\ \text{ð}\,\text{ð'}\, \eta^{\{w, z\}} - \text{ð'}\,\text{ð}\, \eta^{\{w, z\}} \Big\}$

In[337]:= **Equal[#, # //. GHPToNP] & /@ %**

Out[337]= $\{\text{þ þ' } \eta^{\{w, z\}} - \text{þ' þ } \eta^{\{w, z\}} == \eta^{\{w, z\}} (-w (D\gamma) - z (D\overline{\gamma})) + (-w\gamma - z\overline{\gamma}) (D\eta^{\{w, z\}}) -$

$(-(1+w)\gamma - (1+z)\overline{\gamma}) ((-w\epsilon - z\overline{\epsilon})\eta^{\{w, z\}} + D\eta^{\{w, z\}}) + D\triangle\eta^{\{w, z\}} - \eta^{\{w, z\}} (-w(\triangle\epsilon) - z(\triangle\overline{\epsilon})) -$

$(-w\epsilon - z\overline{\epsilon})(\triangle\eta^{\{w, z\}}) + (-(-1+w)\epsilon - (-1+z)\overline{\epsilon})((-w\gamma - z\overline{\gamma})\eta^{\{w, z\}} + \triangle\eta^{\{w, z\}}) - \triangle D\eta^{\{w, z\}},$

$\text{þ ð } \eta^{\{w, z\}} - \text{ð þ } \eta^{\{w, z\}} == \eta^{\{w, z\}} (-z(D\overline{\alpha}) - w(D\beta)) + (-z\overline{\alpha} - w\beta)(D\eta^{\{w, z\}}) -$

$(-(1+z)\overline{\alpha} - (1+w)\beta)((-w\epsilon - z\overline{\epsilon})\eta^{\{w, z\}} + D\eta^{\{w, z\}}) + D\delta\eta^{\{w, z\}} - \eta^{\{w, z\}} (-w(\delta\epsilon) - z(\delta\overline{\epsilon})) -$

$(-w\epsilon - z\overline{\epsilon})(\delta\eta^{\{w, z\}}) + ((1+w)\epsilon - (-1+z)\overline{\epsilon})((-z\overline{\alpha} - w\beta)\eta^{\{w, z\}} + \delta\eta^{\{w, z\}}) - \delta D\eta^{\{w, z\}},$

$\text{þ' ð } \eta^{\{w, z\}} - \text{ð þ' } \eta^{\{w, z\}} == \eta^{\{w, z\}} (-z(\triangle\overline{\alpha}) - w(\triangle\beta)) + (-z\overline{\alpha} - w\beta)(\triangle\eta^{\{w, z\}}) -$

$(-(-1+z)\overline{\alpha} - (-1+w)\beta)((-w\gamma - z\overline{\gamma})\eta^{\{w, z\}} + \triangle\eta^{\{w, z\}}) +$

$\triangle\delta\eta^{\{w, z\}} - \eta^{\{w, z\}} (-w(\delta\gamma) - z(\delta\overline{\gamma})) - (-w\gamma - z\overline{\gamma})(\delta\eta^{\{w, z\}}) +$

$(-(1+w)\gamma - (-1+z)\overline{\gamma})((-z\overline{\alpha} - w\beta)\eta^{\{w, z\}} + \delta\eta^{\{w, z\}}) - \delta\triangle\eta^{\{w, z\}},$

$\text{þ ð' } \eta^{\{w, z\}} - \text{ð' þ } \eta^{\{w, z\}} == \eta^{\{w, z\}} (-w(D\alpha) - z(D\overline{\beta})) + (-w\alpha - z\overline{\beta})(D\eta^{\{w, z\}}) -$

$(-(1+w)\alpha - (1+z)\overline{\beta})((-w\epsilon - z\overline{\epsilon})\eta^{\{w, z\}} + D\eta^{\{w, z\}}) + D\overline{\delta}\eta^{\{w, z\}} - \eta^{\{w, z\}} (-w(\overline{\delta}\epsilon) - z(\overline{\delta}\overline{\epsilon})) -$

$(-w\epsilon - z\overline{\epsilon})(\overline{\delta}\eta^{\{w, z\}}) + (-(-1+w)\epsilon - (1+z)\overline{\epsilon})((-w\alpha - z\overline{\beta})\eta^{\{w, z\}} + \overline{\delta}\eta^{\{w, z\}}) - \overline{\delta}D\eta^{\{w, z\}},$

$\text{þ' ð' } \eta^{\{w, z\}} - \text{ð' þ' } \eta^{\{w, z\}} == \eta^{\{w, z\}} (-w(\triangle\alpha) - z(\triangle\overline{\beta})) + (-w\alpha - z\overline{\beta})(\triangle\eta^{\{w, z\}}) -$

$(-(-1+w)\alpha - (-1+z)\overline{\beta})((-w\gamma - z\overline{\gamma})\eta^{\{w, z\}} + \triangle\eta^{\{w, z\}}) +$

$\triangle\overline{\delta}\eta^{\{w, z\}} - \eta^{\{w, z\}} (-w(\overline{\delta}\gamma) - z(\overline{\delta}\overline{\gamma})) - (-w\gamma - z\overline{\gamma})(\overline{\delta}\eta^{\{w, z\}}) +$

$(-(-1+w)\gamma - (1+z)\overline{\gamma})((-w\alpha - z\overline{\beta})\eta^{\{w, z\}} + \overline{\delta}\eta^{\{w, z\}}) - \overline{\delta}\triangle\eta^{\{w, z\}},$

$\text{ð ð' } \eta^{\{w, z\}} - \text{ð' ð } \eta^{\{w, z\}} == \eta^{\{w, z\}} (-w(\delta\alpha) - z(\delta\overline{\beta})) + (-w\alpha - z\overline{\beta})(\delta\eta^{\{w, z\}}) -$

$(-(1+w)\alpha - (-1+z)\overline{\beta})((-z\overline{\alpha} - w\beta)\eta^{\{w, z\}} + \delta\eta^{\{w, z\}}) +$

$\delta\overline{\delta}\eta^{\{w, z\}} - \eta^{\{w, z\}} (-z(\overline{\delta}\overline{\alpha}) - w(\overline{\delta}\beta)) - (-z\overline{\alpha} - w\beta)(\overline{\delta}\eta^{\{w, z\}}) +$

$(-(1+z)\overline{\alpha} - (-1+w)\beta)((-w\alpha - z\overline{\beta})\eta^{\{w, z\}} + \overline{\delta}\eta^{\{w, z\}}) - \overline{\delta}\delta\eta^{\{w, z\}}\}$

In[338]:= **% /. (NPCommutators /. ψ[] -> η[LI@{w, z}] /. Equal -> Rule);**

In[339]:= **% /. (Map[Dagger, NPCommutators, {2}] /. ψ[] -> η[LI@{w, z}] /. Equal -> Rule);**

In[340]:= **Map[FullSimplify, %, {2}]**

Out[340]= $\{\text{þ þ' } \eta^{\{w, z\}} - \text{þ' þ } \eta^{\{w, z\}} == -\eta^{\{w, z\}}$

$(2w\gamma\epsilon + w\overline{\gamma}\epsilon + z\overline{\gamma}\epsilon + w\gamma\overline{\epsilon} + z\gamma\overline{\epsilon} + 2z\overline{\gamma}\overline{\epsilon} + w(D\gamma) + z(D\overline{\gamma}) - w(\triangle\epsilon) - z(\triangle\overline{\epsilon})) +$

$(\pi + \overline{\tau})(\delta\eta^{\{w, z\}}) + (\overline{\pi} + \tau)(\overline{\delta}\eta^{\{w, z\}}), \text{þ ð } \eta^{\{w, z\}} - \text{ð þ } \eta^{\{w, z\}} == \overline{\pi}(D\eta^{\{w, z\}}) -$

$\kappa(\triangle\eta^{\{w, z\}}) - \eta^{\{w, z\}} ((w-z)\overline{\alpha}\epsilon + (2z\overline{\alpha} + (w+z)\beta)\overline{\epsilon} + z(D\overline{\alpha}) + w(D\beta - \delta\epsilon) - z(\delta\overline{\epsilon})) +$

$\overline{\rho}(\delta\eta^{\{w, z\}}) + \sigma(\overline{\delta}\eta^{\{w, z\}}), \text{þ' ð } \eta^{\{w, z\}} - \text{ð þ' } \eta^{\{w, z\}} == \overline{\nu}(D\eta^{\{w, z\}}) - \tau(\triangle\eta^{\{w, z\}}) +$

$\eta^{\{w, z\}} (w\overline{\alpha}\gamma + z\overline{\alpha}\gamma + 2w\beta\gamma - w\beta\overline{\gamma} + z\beta\overline{\gamma} - z(\triangle\overline{\alpha}) - w(\triangle\beta) + w(\delta\gamma) + z(\delta\overline{\gamma})) -$

$\mu(\delta\eta^{\{w, z\}}) - \overline{\lambda}(\overline{\delta}\eta^{\{w, z\}}),$

$\text{þ ð' } \eta^{\{w, z\}} - \text{ð' þ } \eta^{\{w, z\}} == \pi(D\eta^{\{w, z\}}) - \overline{\kappa}(\triangle\eta^{\{w, z\}}) + \overline{\sigma}(\delta\eta^{\{w, z\}}) - \eta^{\{w, z\}}$

$(2w\alpha\epsilon + w\overline{\beta}\epsilon + z\overline{\beta}\epsilon - w\alpha\overline{\epsilon} + z\alpha\overline{\epsilon} + w(D\alpha) + z(D\overline{\beta}) - w(\overline{\delta}\epsilon) - z(\overline{\delta}\overline{\epsilon})) + \rho(\overline{\delta}\eta^{\{w, z\}}),$

$\text{þ' ð' } \eta^{\{w, z\}} - \text{ð' þ' } \eta^{\{w, z\}} == \nu(D\eta^{\{w, z\}}) - \overline{\tau}(\triangle\eta^{\{w, z\}}) - \lambda(\delta\eta^{\{w, z\}}) +$

$\eta^{\{w, z\}} ((w-z)\overline{\beta}\gamma + ((w+z)\alpha + 2z\overline{\beta})\overline{\gamma} - w(\triangle\alpha) + w(\overline{\delta}\gamma) + z(-(\triangle\overline{\beta}) + \overline{\delta}\overline{\gamma})) - \overline{\mu}(\overline{\delta}\eta^{\{w, z\}}),$

$\text{ð ð' } \eta^{\{w, z\}} - \text{ð' ð } \eta^{\{w, z\}} == (\mu - \overline{\mu})(D\eta^{\{w, z\}}) + (\rho - \overline{\rho})(\triangle\eta^{\{w, z\}}) +$

$\eta^{\{w, z\}} (w(\alpha(\overline{\alpha} - 2\beta) + \beta\overline{\beta}) - z(\alpha\overline{\alpha} + (-2\overline{\alpha} + \beta)\overline{\beta}) + z(-(\delta\overline{\beta}) + \overline{\delta}\overline{\alpha}) + w(-(\delta\alpha) + \overline{\delta}\beta))\}$

We use the Newman-Penrose equations on this relation.

In[341]:= **% /. (NPRicci /. Equal $\rightarrow$ Rule);**

In[342]:= **% /. (NPRicci† /. Equal $\rightarrow$ Rule);**

In[343]:= `Map[Simplify, %, {2}]`

Out[343]= $\{ \text{þ} \, \text{þ}' \, \eta^{\{w, \, z\}} - \text{þ}' \, \text{þ} \, \eta^{\{w, \, z\}} == -\eta^{\{w, \, z\}} \left(w \left(\Lambda - \Phi_{11} + \Psi_2 - \kappa \, \nu + \beta \, \pi + \alpha \, \overline{\pi} + \alpha \, \tau + \pi \, \tau + \beta \, \overline{\tau} \right) + \right.$
$\left. z \left(\Lambda - \Phi_{11} + \overline{\Psi}_2 - \overline{\kappa} \, \overline{\nu} + \overline{\alpha} \, \pi + \overline{\beta} \, \overline{\pi} + \overline{\beta} \, \tau + \overline{\alpha} \, \overline{\tau} + \overline{\pi} \, \overline{\tau} \right) \right) +$
$\left(\pi + \overline{\tau} \right) \left(\delta \, \eta^{\{w, \, z\}} \right) + \left(\overline{\pi} + \tau \right) \left(\overline{\delta} \, \eta^{\{w, \, z\}} \right), \, \text{þ} \, \delta \, \eta^{\{w, \, z\}} - \delta \, \text{þ} \, \eta^{\{w, \, z\}} ==$
$-\eta^{\{w, \, z\}} \left(z \left(-\Phi_{01} - \overline{\gamma} \, \kappa - \overline{\kappa} \, \overline{\lambda} + \overline{\epsilon} \, \overline{\pi} + \overline{\alpha} \, \overline{\rho} + \overline{\pi} \, \overline{\rho} + \overline{\beta} \, \sigma \right) + \right.$
$\left. w \left(\Psi_1 - \gamma \, \kappa - \kappa \, \mu + \epsilon \, \overline{\pi} + \beta \, \overline{\rho} + \alpha \, \sigma + \pi \, \sigma \right) \right) +$
$\overline{\pi} \left(D \, \eta^{\{w, \, z\}} \right) - \kappa \left(\Delta \, \eta^{\{w, \, z\}} \right) + \overline{\rho} \left(\delta \, \eta^{\{w, \, z\}} \right) + \sigma \left(\overline{\delta} \, \eta^{\{w, \, z\}} \right),$
$\text{þ}' \, \delta \, \eta^{\{w, \, z\}} - \delta \, \text{þ}' \, \eta^{\{w, \, z\}} == \eta^{\{w, \, z\}} \left(w \left(-\Phi_{12} + \alpha \, \overline{\lambda} + \beta \, \mu - \epsilon \, \overline{\nu} - \nu \, \sigma + \gamma \, \tau + \mu \, \tau \right) + \right.$
$\left. z \left(\overline{\Psi}_3 + \overline{\beta} \, \overline{\lambda} + \overline{\alpha} \, \mu - \overline{\epsilon} \, \overline{\nu} - \overline{\nu} \, \overline{\rho} + \overline{\gamma} \, \tau + \overline{\lambda} \, \overline{\tau} \right) \right) +$
$\overline{\nu} \left(D \, \eta^{\{w, \, z\}} \right) - \tau \left(\Delta \, \eta^{\{w, \, z\}} \right) - \mu \left(\delta \, \eta^{\{w, \, z\}} \right) - \overline{\lambda} \left(\overline{\delta} \, \eta^{\{w, \, z\}} \right),$
$\text{þ} \, \delta' \, \eta^{\{w, \, z\}} - \delta' \, \text{þ} \, \eta^{\{w, \, z\}} == \eta^{\{w, \, z\}} \left(w \left(\Phi_{10} + \gamma \, \overline{\kappa} + \kappa \, \lambda - \epsilon \, \pi - \alpha \, \rho - \pi \, \rho - \beta \, \overline{\sigma} \right) - \right.$
$\left. z \left(\overline{\Psi}_1 - \overline{\gamma} \, \overline{\kappa} - \overline{\kappa} \, \overline{\mu} + \overline{\epsilon} \, \pi + \overline{\beta} \, \rho + \overline{\alpha} \, \overline{\sigma} + \overline{\pi} \, \overline{\sigma} \right) \right) +$
$\pi \left(D \, \eta^{\{w, \, z\}} \right) - \overline{\kappa} \left(\Delta \, \eta^{\{w, \, z\}} \right) + \overline{\sigma} \left(\delta \, \eta^{\{w, \, z\}} \right) + \rho \left(\overline{\delta} \, \eta^{\{w, \, z\}} \right),$
$\text{þ}' \, \delta' \, \eta^{\{w, \, z\}} - \delta' \, \text{þ}' \, \eta^{\{w, \, z\}} == \eta^{\{w, \, z\}} \left(w \left(\Psi_3 + \beta \, \lambda + \alpha \, \overline{\mu} - \epsilon \, \nu - \nu \, \rho + \lambda \, \tau + \gamma \, \overline{\tau} \right) + \right.$
$\left. z \left(-\Phi_{21} + \overline{\alpha} \, \lambda + \overline{\beta} \, \overline{\mu} - \overline{\epsilon} \, \nu - \overline{\nu} \, \overline{\sigma} + \overline{\gamma} \, \overline{\tau} + \overline{\mu} \, \overline{\tau} \right) \right) +$
$\nu \left(D \, \eta^{\{w, \, z\}} \right) - \overline{\tau} \left(\Delta \, \eta^{\{w, \, z\}} \right) - \lambda \left(\delta \, \eta^{\{w, \, z\}} \right) - \overline{\mu} \left(\overline{\delta} \, \eta^{\{w, \, z\}} \right),$
$\delta \, \delta' \, \eta^{\{w, \, z\}} - \delta' \, \delta \, \eta^{\{w, \, z\}} == \eta^{\{w, \, z\}} \left(w \left(\Lambda + \Phi_{11} + \Psi_2 - \epsilon \, \mu + \epsilon \, \overline{\mu} - \gamma \, \rho - \mu \, \rho + \gamma \, \overline{\rho} + \lambda \, \sigma \right) - \right.$
$\left. z \left(\Lambda + \Phi_{11} + \overline{\Psi}_2 + \overline{\epsilon} \, \mu - \overline{\epsilon} \, \overline{\mu} + \overline{\gamma} \, \rho - \overline{\gamma} \, \overline{\rho} - \overline{\mu} \, \overline{\rho} + \overline{\lambda} \, \sigma \right) \right) +$
$\left(\mu - \overline{\mu} \right) \left(D \, \eta^{\{w, \, z\}} \right) + \left(\rho - \overline{\rho} \right) \left(\Delta \, \eta^{\{w, \, z\}} \right) \}$

Finally we write $\delta \, \eta^{\{w, \, z\}}$, $\overline{\delta} \, \eta^{\{w, \, z\}}$ in terms of $\delta \, \eta^{\{w, \, z\}}$, $\delta' \, \eta^{\{w, \, z\}}$ and simplify the resulting expression.

In[344]:= `% /. NPToGHP;`

In[345]:= `Map[FullSimplify, %, {2}]`

Out[345]= $\{ \text{þ} \, \text{þ}' \, \eta^{\{w, \, z\}} - \text{þ}' \, \text{þ} \, \eta^{\{w, \, z\}} == -\eta^{\{w, \, z\}} \left(w \left(\Lambda - \Phi_{11} + \Psi_2 - \kappa \, \nu + \pi \, \tau \right) + z \left(\Lambda - \Phi_{11} + \overline{\Psi}_2 - \overline{\kappa} \, \overline{\nu} + \overline{\pi} \, \overline{\tau} \right) \right) +$
$\left(\pi + \overline{\tau} \right) \left(\delta \, \eta^{\{w, \, z\}} \right) + \left(\overline{\pi} + \tau \right) \left(\delta' \, \eta^{\{w, \, z\}} \right),$
$\text{þ} \, \delta \, \eta^{\{w, \, z\}} - \delta \, \text{þ} \, \eta^{\{w, \, z\}} == \eta^{\{w, \, z\}} \left(z \left(\Phi_{01} + \overline{\kappa} \, \overline{\lambda} - \overline{\pi} \, \overline{\rho} \right) - w \left(\Psi_1 - \kappa \, \mu + \pi \, \sigma \right) \right) +$
$\overline{\pi} \left(\text{þ} \, \eta^{\{w, \, z\}} \right) - \kappa \left(\text{þ}' \, \eta^{\{w, \, z\}} \right) + \overline{\rho} \left(\delta \, \eta^{\{w, \, z\}} \right) + \sigma \left(\delta' \, \eta^{\{w, \, z\}} \right),$
$\text{þ}' \, \delta \, \eta^{\{w, \, z\}} - \delta \, \text{þ}' \, \eta^{\{w, \, z\}} == \eta^{\{w, \, z\}} \left(-w \left(\Phi_{12} + \nu \, \sigma - \mu \, \tau \right) + z \left(\overline{\Psi}_3 - \overline{\nu} \, \overline{\rho} + \overline{\lambda} \, \overline{\tau} \right) \right) +$
$\overline{\nu} \left(\text{þ} \, \eta^{\{w, \, z\}} \right) - \tau \left(\text{þ}' \, \eta^{\{w, \, z\}} \right) - \mu \left(\delta \, \eta^{\{w, \, z\}} \right) - \overline{\lambda} \left(\delta' \, \eta^{\{w, \, z\}} \right),$
$\text{þ} \, \delta' \, \eta^{\{w, \, z\}} - \delta' \, \text{þ} \, \eta^{\{w, \, z\}} == \eta^{\{w, \, z\}} \left(w \left(\Phi_{10} + \kappa \, \lambda - \pi \, \rho \right) - z \left(\overline{\Psi}_1 - \overline{\kappa} \, \overline{\mu} + \overline{\pi} \, \overline{\sigma} \right) \right) +$
$\pi \left(\text{þ} \, \eta^{\{w, \, z\}} \right) - \overline{\kappa} \left(\text{þ}' \, \eta^{\{w, \, z\}} \right) + \overline{\sigma} \left(\delta \, \eta^{\{w, \, z\}} \right) + \rho \left(\delta' \, \eta^{\{w, \, z\}} \right),$
$\text{þ}' \, \delta' \, \eta^{\{w, \, z\}} - \delta' \, \text{þ}' \, \eta^{\{w, \, z\}} == \eta^{\{w, \, z\}} \left(w \left(\Psi_3 - \nu \, \rho + \lambda \, \tau \right) - z \left(\Phi_{21} + \overline{\nu} \, \overline{\sigma} - \overline{\mu} \, \overline{\tau} \right) \right) +$
$\nu \left(\text{þ} \, \eta^{\{w, \, z\}} \right) - \overline{\tau} \left(\text{þ}' \, \eta^{\{w, \, z\}} \right) - \lambda \left(\delta \, \eta^{\{w, \, z\}} \right) - \overline{\mu} \left(\delta' \, \eta^{\{w, \, z\}} \right),$
$\delta \, \delta' \, \eta^{\{w, \, z\}} - \delta' \, \delta \, \eta^{\{w, \, z\}} == \eta^{\{w, \, z\}} \left(w \left(\Lambda + \Phi_{11} + \Psi_2 - \mu \, \rho + \lambda \, \sigma \right) - z \left(\Lambda + \Phi_{11} + \overline{\Psi}_2 - \overline{\mu} \, \overline{\rho} + \overline{\lambda} \, \sigma \right) \right) +$
$\left(\mu - \overline{\mu} \right) \left(\text{þ} \, \eta^{\{w, \, z\}} \right) + \left(\rho - \overline{\rho} \right) \left(\text{þ}' \, \eta^{\{w, \, z\}} \right) \}$

In[346]:= `GHPCommutators = %;`

We check that everything is OK

In[347]:= `Map[GHPWeightOf, GHPCommutators, {2}]`

Out[347]= `{True, True, True, True, True, True}`

■ **Ricci identities in the GHP formalism**

In this subsection we use the Newman-Penrose Ricci identities (Newman-Penrose equations) to find a set of equations which only contain the GHP differential operators and spin weighted quantities. This set of equations is known as GHP equations or compacted spin coefficient equations.

These are the weighted spin coefficients.

In[348]:= **GHPSpinCoefficients**

Out[348]= $\{\kappa, \lambda, \mu, \nu, \pi, \rho, \sigma, \tau\}$

These are the NP differentials of these spin coefficients.

In[349]:= **Flatten@Outer[PDNP[#1][#2] &, {{1, -NP}, {2, -NP}, {3, -NP}, {4, -NP}}, GHPSpinCoefficients, 1]**

Out[349]= $\{D\kappa, D\lambda, D\mu, D\nu, D\pi, D\rho, D\sigma, D\tau, \Delta\kappa, \Delta\lambda, \Delta\mu, \Delta\nu, \Delta\pi, \Delta\rho, \Delta\sigma, \Delta\tau,$
$\delta\kappa, \delta\lambda, \delta\mu, \delta\nu, \delta\pi, \delta\rho, \delta\sigma, \delta\tau, \overline{\delta}\kappa, \overline{\delta}\lambda, \overline{\delta}\mu, \overline{\delta}\nu, \overline{\delta}\pi, \overline{\delta}\rho, \overline{\delta}\sigma, \overline{\delta}\tau\}$

We compute the GHP differentials of the weighted Newman-Penrose spin coefficients.

In[350]:= **Rule[#, # /. NPToGHP] & /@ %**

Out[350]= $\{D\kappa \to 3\epsilon\kappa + \overline{\epsilon}\kappa + \text{þ}\kappa, D\lambda \to -3\epsilon\lambda + \overline{\epsilon}\lambda + \text{þ}\lambda,$
$D\mu \to -\epsilon\mu - \overline{\epsilon}\mu + \text{þ}\mu, D\nu \to -3\epsilon\nu - \overline{\epsilon}\nu + \text{þ}\nu, D\pi \to -\epsilon\pi + \overline{\epsilon}\pi + \text{þ}\pi,$
$D\rho \to \epsilon\rho + \overline{\epsilon}\rho + \text{þ}\rho, D\sigma \to 3\epsilon\sigma - \overline{\epsilon}\sigma + \text{þ}\sigma, D\tau \to \epsilon\tau - \overline{\epsilon}\tau + \text{þ}\tau,$
$\Delta\kappa \to 3\gamma\kappa + \overline{\gamma}\kappa + \text{þ}'\kappa, \Delta\lambda \to -3\gamma\lambda + \overline{\gamma}\lambda + \text{þ}'\lambda, \Delta\mu \to -\gamma\mu - \overline{\gamma}\mu + \text{þ}'\mu,$
$\Delta\nu \to -3\gamma\nu - \overline{\gamma}\nu + \text{þ}'\nu, \Delta\pi \to -\gamma\pi + \overline{\gamma}\pi + \text{þ}'\pi, \Delta\rho \to \gamma\rho + \overline{\gamma}\rho + \text{þ}'\rho,$
$\Delta\sigma \to 3\gamma\sigma - \overline{\gamma}\sigma + \text{þ}'\sigma, \Delta\tau \to \gamma\tau - \overline{\gamma}\tau + \text{þ}'\tau, \delta\kappa \to \overline{\alpha}\kappa + 3\beta\kappa + \text{ð}\kappa,$
$\delta\lambda \to \overline{\alpha}\lambda - 3\beta\lambda + \text{ð}\lambda, \delta\mu \to -\overline{\alpha}\mu - \beta\mu + \text{ð}\mu, \delta\nu \to -\overline{\alpha}\nu - 3\beta\nu + \text{ð}\nu,$
$\delta\pi \to \overline{\alpha}\pi - \beta\pi + \text{ð}\pi, \delta\rho \to \overline{\alpha}\rho + \beta\rho + \text{ð}\rho, \delta\sigma \to -\overline{\alpha}\sigma + 3\beta\sigma + \text{ð}\sigma,$
$\delta\tau \to -\overline{\alpha}\tau + \beta\tau + \text{ð}\tau, \overline{\delta}\kappa \to 3\alpha\kappa + \overline{\beta}\kappa + \text{ð}'\kappa, \overline{\delta}\lambda \to -3\alpha\lambda + \overline{\beta}\lambda + \text{ð}'\lambda,$
$\overline{\delta}\mu \to -\alpha\mu - \overline{\beta}\mu + \text{ð}'\mu, \overline{\delta}\nu \to -3\alpha\nu - \overline{\beta}\nu + \text{ð}'\nu, \overline{\delta}\pi \to -\alpha\pi + \overline{\beta}\pi + \text{ð}'\pi,$
$\overline{\delta}\rho \to \alpha\rho + \overline{\beta}\rho + \text{ð}'\rho, \overline{\delta}\sigma \to 3\alpha\sigma - \overline{\beta}\sigma + \text{ð}'\sigma, \overline{\delta}\tau \to \alpha\tau - \overline{\beta}\tau + \text{ð}'\tau\}$

With this information we can render the Newman-Penrose equations in terms of the GHP differential operators.

In[351]:= **NPRicci /. %**

Out[351]= $\{ D\,\alpha \;==\; -\Phi_{10} \;-2\,\alpha\,\epsilon \;-\overline{\beta}\,\epsilon \;+\alpha\,\overline{\epsilon} \;-\gamma\,\overline{\kappa} \;-\kappa\,\lambda \;+\epsilon\,\pi \;+\alpha\,\rho \;+\pi\,\rho \;+\beta\,\overline{\sigma} \;+\overline{\delta}\epsilon \;,$

$\quad D\,\beta \;==\; \Psi_1 \;-\overline{\alpha}\,\epsilon \;-\beta\,\overline{\epsilon} \;-\gamma\,\kappa \;-\kappa\,\mu \;+\epsilon\,\overline{\pi} \;+\beta\,\overline{\rho} \;+\alpha\,\sigma \;+\pi\,\sigma \;+\delta\epsilon \;,$

$\quad D\,\gamma \;==\; \Lambda \;-\Phi_{11} \;+\Psi_2 \;-2\,\gamma\,\epsilon \;-\overline{\gamma}\,\epsilon \;-\gamma\,\overline{\epsilon} \;-\kappa\,\nu \;+\beta\,\pi \;+\alpha\,\overline{\pi} \;+\alpha\,\tau \;+\pi\,\tau \;+\beta\,\overline{\tau} \;+\Delta\epsilon \;,$

$\quad -3\,\epsilon\,\lambda \;+\overline{\epsilon}\,\lambda \;+\text{þ}\,\lambda \;==\; -\Phi_{20} \;-3\,\epsilon\,\lambda \;+\overline{\epsilon}\,\lambda \;-\overline{\kappa}\,\nu \;+\pi^2 \;+\lambda\,\rho \;+\mu\,\overline{\sigma} \;+\text{ð}'\,\pi \;,$

$\quad -\epsilon\,\mu \;-\overline{\epsilon}\,\mu \;+\text{þ}\,\mu \;==\; -2\,\Lambda \;+\Psi_2 \;-\epsilon\,\mu \;-\overline{\epsilon}\,\mu \;-\kappa\,\nu \;+\pi\,\overline{\pi} \;+\mu\,\overline{\rho} \;+\lambda\,\sigma \;+\text{ð}\,\pi \;,$

$\quad -3\,\epsilon\,\nu \;-\overline{\epsilon}\,\nu \;+\text{þ}\,\nu \;==\; -\Phi_{21} \;+\Psi_3 \;-3\,\epsilon\,\nu \;-\overline{\epsilon}\,\nu \;+\mu\,\pi \;+\lambda\,\overline{\pi} \;+\lambda\,\tau \;+\mu\,\overline{\tau} \;+\text{þ}'\,\pi \;,$

$\quad \epsilon\,\rho \;+\overline{\epsilon}\,\rho \;+\text{þ}\,\rho \;==\; -\Phi_{00} \;+\kappa\,\pi \;+\epsilon\,\rho \;+\overline{\epsilon}\,\rho \;+\rho^2 \;+\sigma\,\overline{\sigma} \;-\overline{\kappa}\,\tau \;+\text{ð}'\,\kappa \;,$

$\quad 3\,\epsilon\,\sigma \;-\overline{\epsilon}\,\sigma \;+\text{þ}\,\sigma \;==\; \Psi_0 \;+\kappa\,\overline{\pi} \;+3\,\epsilon\,\sigma \;-\overline{\epsilon}\,\sigma \;+\rho\,\sigma \;+\overline{\rho}\,\sigma \;-\kappa\,\tau \;+\text{ð}\,\kappa \;,$

$\quad \epsilon\,\tau \;-\overline{\epsilon}\,\tau \;+\text{þ}\,\tau \;==\; -\Phi_{01} \;+\Psi_1 \;+\overline{\pi}\,\rho \;+\pi\,\sigma \;+\epsilon\,\tau \;-\overline{\epsilon}\,\tau \;+\rho\,\tau \;+\sigma\,\overline{\tau} \;+\text{þ}'\,\kappa \;,$

$\quad \Delta\,\alpha \;==\; -\Psi_3 \;+\overline{\beta}\,\gamma \;+\alpha\,\overline{\gamma} \;-\beta\,\lambda \;-\alpha\,\overline{\mu} \;+\epsilon\,\nu \;+\nu\,\rho \;-\lambda\,\tau \;-\gamma\,\overline{\tau} \;+\overline{\delta}\,\gamma \;,$

$\quad \Delta\,\beta \;==\; \Phi_{12} \;+\overline{\alpha}\,\gamma \;+2\,\beta\,\gamma \;-\beta\,\overline{\gamma} \;-\alpha\,\overline{\lambda} \;-\beta\,\mu \;+\epsilon\,\overline{\nu} \;+\nu\,\sigma \;-\gamma\,\tau \;-\mu\,\tau \;+\delta\,\gamma \;,$

$\quad -3\,\gamma\,\lambda \;+\overline{\gamma}\,\lambda \;+\text{þ}'\,\lambda \;==\; -\Psi_4 \;-3\,\gamma\,\lambda \;+\overline{\gamma}\,\lambda \;-\lambda\,\mu \;-\lambda\,\overline{\mu} \;+\nu\,\pi \;-\nu\,\overline{\tau} \;+\text{ð}'\,\nu \;,$

$\quad -\gamma\,\mu \;-\overline{\gamma}\,\mu \;+\text{þ}'\,\mu \;==\; \Phi_{22} \;-\lambda\,\overline{\lambda} \;-\gamma\,\mu \;-\overline{\gamma}\,\mu \;-\mu^2 \;+\overline{\nu}\,\pi \;-\nu\,\tau \;+\text{ð}\,\nu \;,$

$\quad \gamma\,\rho \;+\overline{\gamma}\,\rho \;+\text{þ}'\,\rho \;==\; 2\,\Lambda \;-\Psi_2 \;+\kappa\,\nu \;+\gamma\,\rho \;+\overline{\gamma}\,\rho \;-\overline{\mu}\,\rho \;-\lambda\,\sigma \;-\tau\,\overline{\tau} \;+\text{ð}'\,\tau \;,$

$\quad 3\,\gamma\,\sigma \;-\overline{\gamma}\,\sigma \;+\text{þ}'\,\sigma \;==\; \Phi_{02} \;+\kappa\,\overline{\nu} \;-\overline{\lambda}\,\rho \;+3\,\gamma\,\sigma \;-\overline{\gamma}\,\sigma \;-\mu\,\sigma \;-\tau^2 \;+\text{ð}\,\tau \;,$

$\quad \delta\,\alpha \;==\; -\Lambda \;-\Phi_{11} \;-\Psi_2 \;+\alpha\,\overline{\alpha} \;-2\,\alpha\,\beta \;+\beta\,\overline{\beta} \;+\epsilon\,\mu \;-\epsilon\,\overline{\mu} \;+\gamma\,\rho \;+\mu\,\rho \;-\gamma\,\overline{\rho} \;-\lambda\,\sigma \;+\overline{\delta}\,\beta \;,$

$\quad \overline{\alpha}\,\lambda \;-3\,\beta\,\lambda \;+\text{ð}\,\lambda \;==\; -\Phi_{21} \;-\Psi_3 \;+\overline{\alpha}\,\lambda \;-3\,\beta\,\lambda \;+\mu\,\pi \;-\overline{\mu}\,\pi \;+\nu\,\rho \;-\nu\,\overline{\rho} \;+\text{þ}'\,\mu \;,$

$\quad \overline{\alpha}\,\rho \;+\beta\,\rho \;+\text{ð}\,\rho \;==\; -\Phi_{01} \;-\Psi_1 \;+\kappa\,\mu \;-\kappa\,\overline{\mu} \;+\overline{\alpha}\,\rho \;+\beta\,\rho \;+\rho\,\tau \;-\overline{\rho}\,\tau \;+\text{þ}'\,\sigma \;\}$

In[352]:= **Simplify //@ %**

Out[352]= $\{ \Phi_{10} \;+2\,\alpha\,\epsilon \;+\overline{\beta}\,\epsilon \;+\gamma\,\overline{\kappa} \;+\kappa\,\lambda \;+D\,\alpha \;==\; \epsilon\,\pi \;+\pi\,\rho \;+\alpha\,(\overline{\epsilon}+\rho) \;+\beta\,\overline{\sigma} \;+\overline{\delta}\epsilon \;,$

$\quad \Psi_1 \;+\epsilon\,\overline{\pi} \;+\beta\,\overline{\rho} \;+\alpha\,\sigma \;+\pi\,\sigma \;+\delta\epsilon \;==\; \overline{\alpha}\,\epsilon \;+\beta\,\overline{\epsilon} \;+\gamma\,\kappa \;+\kappa\,\mu \;+D\,\beta \;,$

$\quad \Lambda \;+\Psi_2 \;+\beta\,\pi \;+\alpha\,\overline{\pi} \;+\alpha\,\tau \;+\pi\,\tau \;+\beta\,\overline{\tau} \;+\Delta\epsilon \;==\; \Phi_{11} \;+2\,\gamma\,\epsilon \;+\overline{\gamma}\,\epsilon \;+\gamma\,\overline{\epsilon} \;+\kappa\,\nu \;+D\,\gamma \;,$

$\quad \Phi_{20} \;+\overline{\kappa}\,\nu \;+\text{þ}\,\lambda \;==\; \pi^2 \;+\lambda\,\rho \;+\mu\,\overline{\sigma} \;+\text{ð}'\,\pi \;, \;\; 2\,\Lambda \;+\kappa\,\nu \;+\text{þ}\,\mu \;==\; \Psi_2 \;+\pi\,\overline{\pi} \;+\mu\,\overline{\rho} \;+\lambda\,\sigma \;+\text{ð}\,\pi \;,$

$\quad \Phi_{21} \;+\text{þ}\,\nu \;==\; \Psi_3 \;+\mu\,\pi \;+\lambda\,\overline{\pi} \;+\lambda\,\tau \;+\mu\,\overline{\tau} \;+\text{þ}'\,\pi \;, \;\; \Phi_{00} \;+\overline{\kappa}\,\tau \;+\text{þ}\,\rho \;==\; \kappa\,\pi \;+\rho^2 \;+\sigma\,\overline{\sigma} \;+\text{ð}'\,\kappa \;,$

$\quad \Psi_0 \;+\kappa\,\overline{\pi} \;+\rho\,\sigma \;+\overline{\rho}\,\sigma \;+\text{ð}\,\kappa \;==\; \kappa\,\tau \;+\text{þ}\,\sigma \;, \;\; \Phi_{01} \;+\text{þ}\,\tau \;==\; \Psi_1 \;+\overline{\pi}\,\rho \;+\pi\,\sigma \;+\rho\,\tau \;+\sigma\,\overline{\tau} \;+\text{þ}'\,\kappa \;,$

$\quad \Psi_3 \;+\beta\,\lambda \;+\alpha\,\overline{\mu} \;+\lambda\,\tau \;+\gamma\,\overline{\tau} \;+\Delta\,\alpha \;==\; \overline{\beta}\,\gamma \;+\alpha\,\overline{\gamma} \;+\epsilon\,\nu \;+\nu\,\rho \;+\overline{\delta}\,\gamma \;,$

$\quad \alpha\,\overline{\lambda} \;+\beta\,(\overline{\gamma}+\mu) \;+\gamma\,\tau \;+\mu\,\tau \;+\Delta\,\beta \;==\; \Phi_{12} \;+\overline{\alpha}\,\gamma \;+2\,\beta\,\gamma \;+\epsilon\,\overline{\nu} \;+\nu\,\sigma \;+\delta\,\gamma \;,$

$\quad \Psi_4 \;+\lambda\,\mu \;+\lambda\,\overline{\mu} \;+\nu\,\overline{\tau} \;+\text{þ}'\,\lambda \;==\; \nu\,\pi \;+\text{ð}'\,\nu \;, \;\; \Phi_{22} \;+\overline{\nu}\,\pi \;+\text{ð}\,\nu \;==\; \lambda\,\overline{\lambda} \;+\mu^2 \;+\nu\,\tau \;+\text{þ}'\,\mu \;,$

$\quad 2\,\Lambda \;+\kappa\,\nu \;+\text{ð}'\,\tau \;==\; \Psi_2 \;+\overline{\mu}\,\rho \;+\lambda\,\sigma \;+\tau\,\overline{\tau} \;+\text{þ}'\,\rho \;, \;\; \Phi_{02} \;+\kappa\,\overline{\nu} \;+\text{ð}\,\tau \;==\; \overline{\lambda}\,\rho \;+\mu\,\sigma \;+\tau^2 \;+\text{þ}'\,\sigma \;,$

$\quad \Lambda \;+\Phi_{11} \;+\Psi_2 \;+2\,\alpha\,\beta \;+\epsilon\,\overline{\mu} \;+\gamma\,\overline{\rho} \;+\lambda\,\sigma \;+\delta\,\alpha \;==\; \alpha\,\overline{\alpha} \;+\beta\,\overline{\beta} \;+\epsilon\,\mu \;+\gamma\,\rho \;+\mu\,\rho \;+\overline{\delta}\,\beta \;,$

$\quad \Phi_{21} \;+\Psi_3 \;+\overline{\mu}\,\pi \;+\nu\,\overline{\rho} \;+\text{ð}\,\lambda \;==\; \mu\,\pi \;+\nu\,\rho \;+\text{þ}'\,\mu \;, \;\; \Phi_{01} \;+\Psi_1 \;+\kappa\,\overline{\mu} \;+\overline{\rho}\,\tau \;+\text{ð}\,\rho \;==\; \kappa\,\mu \;+\rho\,\tau \;+\text{þ}'\,\sigma \;\}$

We are interested in these equations which only contain weighted spin coefficients and GHP operators. These are

In[353]:= **Part[%, Union[First /@ Position[%, Θ_[_]@_]]]**

Out[353]= $\{ \Phi_{20} \;+\overline{\kappa}\,\nu \;+\text{þ}\,\lambda \;==\; \pi^2 \;+\lambda\,\rho \;+\mu\,\overline{\sigma} \;+\text{ð}'\,\pi \;, \;\; 2\,\Lambda \;+\kappa\,\nu \;+\text{þ}\,\mu \;==\; \Psi_2 \;+\pi\,\overline{\pi} \;+\mu\,\overline{\rho} \;+\lambda\,\sigma \;+\text{ð}\,\pi \;,$

$\quad \Phi_{21} \;+\text{þ}\,\nu \;==\; \Psi_3 \;+\mu\,\pi \;+\lambda\,\overline{\pi} \;+\lambda\,\tau \;+\mu\,\overline{\tau} \;+\text{þ}'\,\pi \;, \;\; \Phi_{00} \;+\overline{\kappa}\,\tau \;+\text{þ}\,\rho \;==\; \kappa\,\pi \;+\rho^2 \;+\sigma\,\overline{\sigma} \;+\text{ð}'\,\kappa \;,$

$\quad \Psi_0 \;+\kappa\,\overline{\pi} \;+\rho\,\sigma \;+\overline{\rho}\,\sigma \;+\text{ð}\,\kappa \;==\; \kappa\,\tau \;+\text{þ}\,\sigma \;, \;\; \Phi_{01} \;+\text{þ}\,\tau \;==\; \Psi_1 \;+\overline{\pi}\,\rho \;+\pi\,\sigma \;+\rho\,\tau \;+\sigma\,\overline{\tau} \;+\text{þ}'\,\kappa \;,$

$\quad \Psi_4 \;+\lambda\,\mu \;+\lambda\,\overline{\mu} \;+\nu\,\overline{\tau} \;+\text{þ}'\,\lambda \;==\; \nu\,\pi \;+\text{ð}'\,\nu \;, \;\; \Phi_{22} \;+\overline{\nu}\,\pi \;+\text{ð}\,\nu \;==\; \lambda\,\overline{\lambda} \;+\mu^2 \;+\nu\,\tau \;+\text{þ}'\,\mu \;,$

$\quad 2\,\Lambda \;+\kappa\,\nu \;+\text{ð}'\,\tau \;==\; \Psi_2 \;+\overline{\mu}\,\rho \;+\lambda\,\sigma \;+\tau\,\overline{\tau} \;+\text{þ}'\,\rho \;, \;\; \Phi_{02} \;+\kappa\,\overline{\nu} \;+\text{ð}\,\tau \;==\; \overline{\lambda}\,\rho \;+\mu\,\sigma \;+\tau^2 \;+\text{þ}'\,\sigma \;,$

$\quad \Phi_{21} \;+\Psi_3 \;+\overline{\mu}\,\pi \;+\nu\,\overline{\rho} \;+\text{ð}\,\lambda \;==\; \mu\,\pi \;+\nu\,\rho \;+\text{þ}'\,\mu \;, \;\; \Phi_{01} \;+\Psi_1 \;+\kappa\,\overline{\mu} \;+\overline{\rho}\,\tau \;+\text{ð}\,\rho \;==\; \kappa\,\mu \;+\rho\,\tau \;+\text{þ}'\,\sigma \;\}$

We may write these equations in an alternative way.

In[354]:= **Solve[%, Union@Cases[%, θ[__]@_, Infinity]] // Flatten**

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

Out[354]= $\{ þ\,\lambda \to -\Phi_{20} - \overline{\kappa}\,\nu + \pi^2 + \lambda\,\rho + \mu\,\overline{\sigma} + ð'\pi$, $þ\,\mu \to -2\,\Lambda + \Psi_2 - \kappa\,\nu + \pi\,\overline{\pi} + \mu\,\overline{\rho} + \lambda\,\sigma + ð\,\pi$,
$þ\,\nu \to -\Phi_{21} + \Psi_3 + \mu\,\pi + \lambda\,\overline{\pi} + \lambda\,\tau + \mu\,\overline{\tau} + þ'\pi$,
$þ\,\rho \to -\Phi_{00} + \kappa\,\pi + \rho^2 + \sigma\,\overline{\sigma} - \overline{\kappa}\,\tau + ð'\kappa$, $þ\,\sigma \to \Psi_0 + \kappa\,\overline{\pi} + \rho\,\sigma + \overline{\rho}\,\sigma - \kappa\,\tau + ð\,\kappa$,
$þ\,\tau \to -\Phi_{01} + \Psi_1 + \overline{\pi}\,\rho + \pi\,\sigma + \rho\,\tau + \sigma\,\overline{\tau} + þ'\kappa$, $þ'\lambda \to -\Psi_4 - \lambda\,\mu - \lambda\,\overline{\mu} + \nu\,\pi - \nu\,\overline{\tau} + ð'\nu$,
$þ'\mu \to \Phi_{22} - \lambda\,\overline{\lambda} - \mu^2 + \overline{\nu}\,\pi - \nu\,\tau + ð\,\nu$, $þ'\rho \to 2\,\Lambda - \Psi_2 + \kappa\,\nu - \overline{\mu}\,\rho - \lambda\,\sigma - \tau\,\overline{\tau} + ð'\tau$,
$þ'\sigma \to \Phi_{02} + \kappa\,\overline{\nu} - \overline{\lambda}\,\rho - \mu\,\sigma - \tau^2 + ð\,\tau$, $ð\,\lambda \to -\Phi_{21} - \Psi_3 + \mu\,\pi - \overline{\mu}\,\pi + \nu\,\rho - \nu\,\overline{\rho} + ð'\mu$,
$ð\,\rho \to -\Phi_{01} - \Psi_1 + \kappa\,\mu - \kappa\,\overline{\mu} + \rho\,\tau - \overline{\rho}\,\tau + ð'\sigma \}$

In[355]:= **FullSimplify //@ %**

Out[355]= $\{ þ\,\lambda \to -\Phi_{20} - \overline{\kappa}\,\nu + \pi^2 + \lambda\,\rho + \mu\,\overline{\sigma} + ð'\pi$, $þ\,\mu \to -2\,\Lambda + \Psi_2 - \kappa\,\nu + \pi\,\overline{\pi} + \mu\,\overline{\rho} + \lambda\,\sigma + ð\,\pi$,
$þ\,\nu \to -\Phi_{21} + \Psi_3 + \lambda\,(\overline{\pi} + \tau) + \mu\,(\pi + \overline{\tau}) + þ'\pi$, $þ\,\rho \to -\Phi_{00} + \kappa\,\pi + \rho^2 + \sigma\,\overline{\sigma} - \overline{\kappa}\,\tau + ð'\kappa$,
$þ\,\sigma \to \Psi_0 + (\rho + \overline{\rho})\,\sigma + \kappa\,(\overline{\pi} - \tau) + ð\,\kappa$, $þ\,\tau \to -\Phi_{01} + \Psi_1 + \rho\,(\overline{\pi} + \tau) + \sigma\,(\pi + \overline{\tau}) + þ'\kappa$,
$þ'\lambda \to -\Psi_4 - \lambda\,(\mu + \overline{\mu}) + \nu\,(\pi - \overline{\tau}) + ð'\nu$, $þ'\mu \to \Phi_{22} - \lambda\,\overline{\lambda} - \mu^2 + \overline{\nu}\,\pi - \nu\,\tau + ð\,\nu$,
$þ'\rho \to 2\,\Lambda - \Psi_2 + \kappa\,\nu - \overline{\mu}\,\rho - \lambda\,\sigma - \tau\,\overline{\tau} + ð'\tau$, $þ'\sigma \to \Phi_{02} + \kappa\,\overline{\nu} - \overline{\lambda}\,\rho - \mu\,\sigma - \tau^2 + ð\,\tau$,
$ð\,\lambda \to -\Phi_{21} - \Psi_3 + \mu\,\pi - \overline{\mu}\,\pi + \nu\,\rho - \nu\,\overline{\rho} + ð'\mu$,
$ð\,\rho \to -\Phi_{01} - \Psi_1 + \kappa\,\mu - \kappa\,\overline{\mu} + \rho\,\tau - \overline{\rho}\,\tau + ð'\sigma \}$

In[356]:= **% /. Rule -> Equal**

Out[356]= $\{ þ\,\lambda == -\Phi_{20} - \overline{\kappa}\,\nu + \pi^2 + \lambda\,\rho + \mu\,\overline{\sigma} + ð'\pi$, $þ\,\mu == -2\,\Lambda + \Psi_2 - \kappa\,\nu + \pi\,\overline{\pi} + \mu\,\overline{\rho} + \lambda\,\sigma + ð\,\pi$,
$þ\,\nu == -\Phi_{21} + \Psi_3 + \lambda\,(\overline{\pi} + \tau) + \mu\,(\pi + \overline{\tau}) + þ'\pi$, $þ\,\rho == -\Phi_{00} + \kappa\,\pi + \rho^2 + \sigma\,\overline{\sigma} - \overline{\kappa}\,\tau + ð'\kappa$,
$þ\,\sigma == \Psi_0 + (\rho + \overline{\rho})\,\sigma + \kappa\,(\overline{\pi} - \tau) + ð\,\kappa$, $þ\,\tau == -\Phi_{01} + \Psi_1 + \rho\,(\overline{\pi} + \tau) + \sigma\,(\pi + \overline{\tau}) + þ'\kappa$,
$þ'\lambda == -\Psi_4 - \lambda\,(\mu + \overline{\mu}) + \nu\,(\pi - \overline{\tau}) + ð'\nu$, $þ'\mu == \Phi_{22} - \lambda\,\overline{\lambda} - \mu^2 + \overline{\nu}\,\pi - \nu\,\tau + ð\,\nu$,
$þ'\rho == 2\,\Lambda - \Psi_2 + \kappa\,\nu - \overline{\mu}\,\rho - \lambda\,\sigma - \tau\,\overline{\tau} + ð'\tau$, $þ'\sigma == \Phi_{02} + \kappa\,\overline{\nu} - \overline{\lambda}\,\rho - \mu\,\sigma - \tau^2 + ð\,\tau$,
$ð\,\lambda == -\Phi_{21} - \Psi_3 + \mu\,\pi - \overline{\mu}\,\pi + \nu\,\rho - \nu\,\overline{\rho} + ð'\mu$,
$ð\,\rho == -\Phi_{01} - \Psi_1 + \kappa\,\mu - \kappa\,\overline{\mu} + \rho\,\tau - \overline{\rho}\,\tau + ð'\sigma \}$

Previous set of equations is known as GHP equations or compacted spin coefficient equations.

In[357]:= **Length[GHPRicci = %]**

Out[357]= 12

- **Bianchi identities in the GHP formalism**

Each of the components of the curvature spinors is a weighted scalar. Using this property it is possible to rewrite the Newman-Penrose form of the Bianchi identities in terms of spin weighted quantities. This is accomplished in this subsection.

We write the differentials of the curvature components which appear in the NP Bianchi identities in terms of differentials written in ters of the GHP operators.

In[358]:= **NPBianchi /. NPToGHP**

Out[358]= $\{ 2 \Phi_{01} \epsilon + \text{þ} \Phi_{01} == 4 \Psi_0 \alpha + 2 \Phi_{00} \overline{\alpha} + 2 \Phi_{00} \beta - 2 \Phi_{11} \kappa - 3 \Psi_2 \kappa - \Phi_{02} \overline{\kappa} + \Psi_0 (-4 \alpha + \pi) +$
$\Phi_{00} (-2 (\overline{\alpha} + \beta) + \overline{\pi}) + 4 \Psi_1 \rho + 2 \Phi_{01} (\epsilon + \overline{\rho}) + 2 \Phi_{10} \sigma - \text{þ} \Psi_1 + \text{ð} \Phi_{00} + \text{ð}' \Psi_0 ,$

$2 \Phi_{02} \epsilon - 2 \Phi_{02} \overline{\epsilon} + \text{þ} \Phi_{02} == 4 \Psi_0 \gamma - 2 \Phi_{12} \kappa - \Phi_{00} \overline{\lambda} + \Psi_0 (-4 \gamma + \mu) + 2 \Phi_{01} \overline{\pi} +$
$\Phi_{02} (2 \epsilon - 2 \overline{\epsilon} + \overline{\rho}) + 2 \Phi_{11} \sigma - 3 \Psi_2 \sigma + 4 \Psi_1 \tau + \text{þ}' \Psi_0 + \text{ð} \Phi_{01} - \text{ð} \Psi_1 ,$

$\text{þ} \Phi_{11} == 2 \Psi_1 \alpha + 2 \Phi_{10} \overline{\alpha} - (\Phi_{21} + 2 \Psi_3) \kappa - \Phi_{12} \overline{\kappa} - \Psi_0 \lambda - \Phi_{00} \mu + \Phi_{01} \pi +$
$2 \Psi_1 (-\alpha + \pi) + \Phi_{10} (-2 \overline{\alpha} + \overline{\pi}) + 3 \Psi_2 \rho + 2 \Phi_{11} \overline{\rho} + \Phi_{20} \sigma - \text{þ} \Lambda - \text{þ} \Psi_2 + \text{ð} \Phi_{10} + \text{ð}' \Psi_1 ,$

$-2 \Phi_{22} \epsilon - 2 \Phi_{22} \overline{\epsilon} + \text{þ} \Phi_{22} == -2 \Phi_{21} \beta + 2 \Psi_3 \beta - \Phi_{20} \overline{\lambda} - 2 \Phi_{11} \mu + 3 \Psi_2 \mu - 2 \Psi_1 \nu + 2 \Phi_{12} \pi +$
$2 \Phi_{21} (\beta + \overline{\pi}) + \Phi_{22} (-2 (\epsilon + \overline{\epsilon}) + \overline{\rho}) - \Psi_4 \sigma + 2 \Psi_3 (-\beta + \tau) - 2 (\text{þ}' \Lambda) + \text{þ}' \Psi_2 + \text{ð} \Phi_{21} - \text{ð} \Psi_3 ,$

$-4 \Psi_4 \epsilon + \text{þ} \Psi_4 == -2 \Phi_{20} \gamma + 2 \Phi_{20} \overline{\gamma} - 4 \Psi_4 \epsilon + 2 \Phi_{11} \lambda - 3 \Psi_2 \lambda + \Phi_{20} (2 \gamma - 2 \overline{\gamma} + \overline{\mu}) -$
$2 \Phi_{10} \nu + 4 \Psi_3 \pi + \Psi_4 \rho - \Phi_{22} \overline{\sigma} + 2 \Phi_{21} \overline{\tau} + \text{þ}' \Phi_{20} - \text{ð}' \Phi_{21} + \text{ð}' \Psi_3 , 2 \Phi_{00} \gamma + 2 \Phi_{00} \overline{\gamma} + \text{þ}' \Phi_{00} ==$
$2 \Phi_{01} \alpha - 2 \Psi_1 \alpha + 2 \Phi_{00} (\gamma + \overline{\gamma}) + 2 \Psi_3 \kappa + \Psi_0 \lambda - \Phi_{00} \overline{\mu} + 2 \Psi_1 (\alpha - \pi) + 2 \Phi_{11} \rho -$
$3 \Psi_2 \rho + \Phi_{02} \overline{\sigma} - 2 \Phi_{10} \tau - 2 \Phi_{01} (\alpha + \overline{\tau}) - 2 (\text{þ} \Lambda) + \text{þ} \Psi_2 + \text{ð}' \Phi_{01} - \text{ð}' \Psi_1 ,$

$\text{þ}' \Phi_{11} == -\Phi_{02} \lambda - 3 \Psi_2 \mu - 2 \Phi_{11} \overline{\mu} + \Phi_{01} \nu + 2 \Psi_1 \nu + \Phi_{10} \overline{\nu} + \Phi_{22} \rho + \Psi_4 \sigma -$
$\Phi_{21} \tau - 2 \Psi_3 \tau - \Phi_{12} \overline{\tau} - \text{þ}' \Lambda - \text{þ}' \Psi_2 + \text{ð} \Psi_3 + \text{ð}' \Phi_{12} , -2 \Phi_{21} \gamma + \text{þ}' \Phi_{21} ==$
$-2 \Phi_{22} \alpha - 4 \Psi_4 \beta - 2 \Phi_{22} \overline{\beta} + 2 \Phi_{22} (\alpha + \overline{\beta}) + 2 \Psi_3 \gamma - 2 \Phi_{12} \lambda - 2 (\Psi_3 (\gamma + 2 \mu) + \Phi_{21} (\gamma + \overline{\mu})) +$
$2 \Phi_{11} \nu + 3 \Psi_2 \nu + \Phi_{20} \overline{\nu} + \Psi_4 (4 \beta - \tau) - \Phi_{22} \overline{\tau} - \text{þ}' \Psi_3 + \text{ð} \Psi_4 + \text{ð}' \Phi_{22} ,$

$\text{ð} \Phi_{11} == \Phi_{22} \kappa + \Phi_{10} \overline{\lambda} + \Phi_{01} \mu - 2 \Psi_1 \mu + \Psi_0 \nu - \Phi_{02} \pi - 2 \Phi_{11} \overline{\pi} -$
$\Phi_{12} \overline{\rho} - \Phi_{21} \sigma + 2 \Psi_3 \sigma - 3 \Psi_2 \tau + \text{þ} \Phi_{12} - \text{þ}' \Psi_1 + \text{ð} \Lambda + \text{ð} \Psi_2 ,$

$2 \Phi_{02} \alpha - 2 \Phi_{02} \overline{\beta} + \text{ð}' \Phi_{02} == 2 \Psi_1 \mu + 2 \Phi_{01} \overline{\mu} - \Psi_0 \nu - \Phi_{00} \overline{\nu} - 2 \Phi_{12} \rho - 2 \Psi_3 \sigma +$
$2 \Phi_{11} \tau + 3 \Psi_2 \tau + \Phi_{02} (2 \alpha - 2 \overline{\beta} + \overline{\tau}) + \text{þ}' \Phi_{01} + \text{þ}' \Psi_1 + 2 (\text{ð} \Lambda) - \text{ð} \Psi_2 , 2 (\text{ð}' \Phi_{11}) ==$
$2 \Phi_{20} \overline{\alpha} - 2 \Phi_{20} \beta + 2 \Phi_{21} \epsilon - 6 \Psi_3 \epsilon - 3 \Psi_4 \kappa - \Phi_{22} \overline{\kappa} + 2 \Phi_{01} \lambda - 6 \Psi_1 \lambda - 2 \Phi_{10} (2 \overline{\gamma} + \mu - \overline{\mu}) -$
$2 \Phi_{00} \nu + 2 \Phi_{11} \pi + 9 \Psi_2 \pi + 6 \Psi_3 \rho - 2 \Phi_{21} (\epsilon + \rho - \overline{\rho}) - 2 \Phi_{12} \overline{\sigma} + \Phi_{20} (-2 \overline{\alpha} + 2 \beta + \overline{\pi} + 2 \tau) +$
$4 \Phi_{11} \overline{\tau} - \text{þ} \Phi_{21} - 3 (-2 \Psi_3 \epsilon + \text{þ} \Psi_3) + 2 (2 \Phi_{10} \overline{\gamma} + \text{þ}' \Phi_{10}) + \text{ð} \Phi_{20} + 3 (\text{ð}' \Psi_2) \}$

In[359]:= **Simplify //@ %**

Out[359]= $\{ 2 \Phi_{11} \kappa + 3 \Psi_2 \kappa + \Phi_{02} \overline{\kappa} + \text{þ} \Phi_{01} + \text{þ} \Psi_1 ==$
$\Psi_0 \pi + \Phi_{00} \overline{\pi} + 4 \Psi_1 \rho + 2 \Phi_{01} \overline{\rho} + 2 \Phi_{10} \sigma + \text{ð} \Phi_{00} + \text{ð}' \Psi_0 ,$

$2 \Phi_{12} \kappa + \Phi_{00} \overline{\lambda} + 3 \Psi_2 \sigma + \text{þ} \Phi_{02} + \text{ð} \Psi_1 ==$
$\Psi_0 \mu + 2 \Phi_{01} \overline{\pi} + \Phi_{02} \overline{\rho} + 2 \Phi_{11} \sigma + 4 \Psi_1 \tau + \text{þ}' \Phi_{00} + \text{ð} \Phi_{01} ,$

$\Phi_{21} \kappa + 2 \Psi_3 \kappa + \Phi_{12} \overline{\kappa} + \Psi_0 \lambda + \Phi_{00} \mu + \text{þ} \Lambda + \text{þ} \Phi_{11} + \text{þ} \Psi_2 ==$
$\Phi_{01} \pi + 2 \Psi_1 \pi + \Phi_{10} \overline{\pi} + 3 \Psi_2 \rho + 2 \Phi_{11} \overline{\rho} + \Phi_{20} \sigma + \text{ð} \Phi_{10} + \text{ð}' \Psi_1 ,$

$\Phi_{20} \overline{\lambda} + 2 \Phi_{11} \mu + 2 \Psi_1 \nu + \Psi_4 \sigma + \text{þ} \Phi_{22} + 2 (\text{þ}' \Lambda) + \text{ð} \Psi_3 ==$
$3 \Psi_2 \mu + 2 \Phi_{12} \pi + 2 \Phi_{21} \overline{\pi} + \Phi_{22} \overline{\rho} + 2 \Psi_3 \tau + \text{þ}' \Psi_2 + \text{ð} \Phi_{21} ,$

$2 \Phi_{11} \lambda + \Phi_{20} \overline{\mu} + 4 \Psi_3 \pi + \Psi_4 \rho + 2 \Phi_{21} \overline{\tau} + \text{þ}' \Phi_{20} + \text{ð}' \Psi_3 ==$
$3 \Psi_2 \lambda + 2 \Phi_{10} \nu + \Phi_{22} \overline{\sigma} + \text{þ} \Psi_4 + \text{ð}' \Phi_{21} , 2 \Psi_3 \kappa + \Psi_0 \lambda + 2 \Phi_{11} \rho + \Phi_{02} \overline{\sigma} + \text{þ} \Psi_2 + \text{ð}' \Phi_{01} ==$
$\Phi_{00} \overline{\mu} + 2 \Psi_1 \pi + 3 \Psi_2 \rho + 2 \Phi_{10} \tau + 2 \Phi_{01} \overline{\tau} + 2 (\text{þ} \Lambda) + \text{þ}' \Phi_{00} + \text{ð}' \Psi_1 ,$

$\Phi_{02} \lambda + 3 \Psi_2 \mu + 2 \Phi_{11} \overline{\mu} + \Phi_{21} \tau + 2 \Psi_3 \tau + \Phi_{12} \overline{\tau} + \text{þ}' \Lambda + \text{þ}' \Phi_{11} + \text{þ}' \Psi_2 ==$
$\Phi_{01} \nu + 2 \Psi_1 \nu + \Phi_{10} \overline{\nu} + \Phi_{22} \rho + \Psi_4 \sigma + \text{ð} \Psi_3 + \text{ð}' \Phi_{12} ,$

$2 \Phi_{12} \lambda + 4 \Psi_3 \mu + 2 \Phi_{21} \overline{\mu} + \Psi_4 \tau + \Phi_{22} \overline{\tau} + \text{þ}' \Phi_{21} + \text{þ}' \Psi_3 ==$
$2 \Phi_{11} \nu + 3 \Psi_2 \nu + \Phi_{20} \overline{\nu} + \text{ð} \Psi_4 + \text{ð}' \Phi_{22} ,$

$\Phi_{22} \kappa + \Phi_{10} \overline{\lambda} + \Phi_{01} \mu + \Psi_0 \nu + 2 \Psi_3 \sigma + \text{þ} \Phi_{12} + \text{ð} \Lambda + \text{ð} \Psi_2 ==$
$2 \Psi_1 \mu + \Phi_{02} \pi + 2 \Phi_{11} \overline{\pi} + \Phi_{12} \overline{\rho} + \Phi_{21} \sigma + 3 \Psi_2 \tau + \text{þ}' \Psi_1 + \text{ð} \Phi_{11} ,$

$2 \Psi_1 \mu + 2 \Phi_{01} \overline{\mu} + 2 \Phi_{11} \tau + 3 \Psi_2 \tau + \Phi_{02} \overline{\tau} + \text{þ}' \Phi_{01} + \text{þ}' \Psi_1 + 2 (\text{ð} \Lambda) ==$
$\Psi_0 \nu + \Phi_{00} \overline{\nu} + 2 \Phi_{12} \rho + 2 \Psi_3 \sigma + \text{ð} \Psi_2 + \text{ð}' \Phi_{02} ,$

$3 \Psi_4 \kappa + \Phi_{22} \overline{\kappa} + 6 \Psi_1 \lambda + 2 \Phi_{10} \mu + 2 \Phi_{00} \nu + 2 \Phi_{21} \rho + 2 \Phi_{12} \overline{\sigma} + \text{þ} \Phi_{21} + 3 (\text{þ} \Psi_3) + 2 (\text{ð}' \Phi_{11}) ==$
$2 \Phi_{01} \lambda + 2 \Phi_{10} \overline{\mu} + 2 \Phi_{11} \pi + 9 \Psi_2 \pi + \Phi_{20} \overline{\pi} + 6 \Psi_3 \rho +$
$2 \Phi_{21} \overline{\rho} + 2 \Phi_{20} \tau + 4 \Phi_{11} \overline{\tau} + 2 (\text{þ}' \Phi_{10}) + \text{ð} \Phi_{20} + 3 (\text{ð}' \Psi_2) \}$

We may write these equations in an alternative way.

In[360]:= **Solve[%, Union@Cases[%, θ[__]@_, Infinity]] // Flatten**

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

Out[360]= $\Big\{ \text{þ}\,\Lambda \,\to\, \frac{1}{2}\, (2\,\Psi_3\ \kappa\ +\Psi_0\ \lambda\ -\Phi_{00}\ \overline{\mu}\ -2\,\Psi_1\ \pi\ +2\,\Phi_{11}\ \rho\ -3\,\Psi_2\ \rho\ +\Phi_{02}\ \overline{\sigma}\ -2\,\Phi_{10}\ \tau\ -2\,\Phi_{01}\ \overline{\tau}\ ) +$

$\frac{1}{2}\,(\text{þ}\,\Psi_2\ ) - \frac{1}{2}\,(\text{þ}'\,\Phi_{00}\ ) + \frac{1}{2}\,(\text{ð}'\,\Phi_{01}\ ) - \frac{1}{2}\,(\text{ð}'\,\Psi_1\ ) ,\ \text{þ}\,\Phi_{01}\ \to\, -2\,\Phi_{11}\ \kappa\ -3\,\Psi_2\ \kappa\ -$

$\Phi_{02}\ \overline{\kappa}\ +\Psi_0\ \pi\ +\Phi_{00}\ \overline{\pi}\ +4\,\Psi_1\ \rho\ +2\,\Phi_{01}\ \overline{\rho}\ +2\,\Phi_{10}\ \sigma\ -\text{þ}\,\Psi_1\ +\text{ð}\,\Phi_{00}\ +\text{ð}'\,\Psi_0\ ,\ \text{þ}\,\Phi_{02}\ \to$

$-2\,\Phi_{12}\ \kappa\ -\Phi_{00}\ \overline{\lambda}\ +\Psi_0\ \mu\ +2\,\Phi_{01}\ \overline{\pi}\ +\Phi_{02}\ \overline{\rho}\ +2\,\Phi_{11}\ \sigma\ -3\,\Psi_2\ \sigma\ +4\,\Psi_1\ \tau\ +\text{þ}'\,\Psi_0\ +\text{ð}\,\Phi_{01}\ -\text{ð}\,\Psi_1\ ,$

$\text{þ}\,\Phi_{11}\ \to\, \frac{1}{2}\, (-2\,\Phi_{21}\ \kappa\ -6\,\Psi_3\ \kappa\ -2\,\Phi_{12}\ \overline{\kappa}\ -3\,\Psi_0\ \lambda\ -2\,\Phi_{00}\ \mu\ +\Phi_{00}\ \overline{\mu}\ +2\,\Phi_{01}\ \pi\ +6\,\Psi_1\ \pi\ +$

$2\,\Phi_{10}\ \overline{\pi}\ -2\,\Phi_{11}\ \rho\ +9\,\Psi_2\ \rho\ +4\,\Phi_{11}\ \overline{\rho}\ +2\,\Phi_{20}\ \sigma\ -\Phi_{02}\ \overline{\sigma}\ +2\,\Phi_{10}\ \tau\ +2\,\Phi_{01}\ \overline{\tau}\ ) -$

$\frac{3}{2}\,(\text{þ}\,\Psi_2\ ) + \frac{1}{2}\,(\text{þ}'\,\Phi_{00}\ ) +\text{ð}\,\Phi_{10}\ - \frac{1}{2}\,(\text{ð}'\,\Phi_{01}\ ) + \frac{3}{2}\,(\text{ð}'\,\Psi_1\ ) ,$

$\text{þ}\,\Phi_{12}\ \to\, -\Phi_{22}\ \kappa\ -\Phi_{10}\ \overline{\lambda}\ -\Phi_{01}\ \mu\ +2\,\Psi_1\ \mu\ -\Psi_0\ \nu\ +\Phi_{02}\ \pi\ +2\,\Phi_{11}\ \overline{\pi}\ +$

$\Phi_{12}\ \overline{\rho}\ +\Phi_{21}\ \sigma\ -2\,\Psi_3\ \sigma\ +3\,\Psi_2\ \tau\ +\text{þ}'\,\Psi_1\ -\text{ð}\,\Lambda\ +\text{ð}\,\Phi_{11}\ -\text{ð}\,\Psi_2\ ,$

$\text{þ}\,\Phi_{21}\ \to\, -3\,\Psi_4\ \kappa\ -\Phi_{22}\ \overline{\kappa}\ +2\,\Phi_{01}\ \lambda\ -6\,\Psi_1\ \lambda\ -2\,\Phi_{10}\ \mu\ +2\,\Phi_{10}\ \overline{\mu}\ -2\,\Phi_{00}\ \nu\ +$

$2\,\Phi_{11}\ \pi\ +9\,\Psi_2\ \pi\ +\Phi_{20}\ \overline{\pi}\ -2\,\Phi_{21}\ \rho\ +6\,\Psi_3\ \rho\ +2\,\Phi_{21}\ \overline{\rho}\ -2\,\Phi_{12}\ \overline{\sigma}\ +$

$2\,\Phi_{20}\ \tau\ +4\,\Phi_{11}\ \overline{\tau}\ -3\,(\text{þ}\,\Psi_3\ ) +2\,(\text{þ}'\,\Phi_{10}\ ) +\text{ð}\,\Phi_{20}\ -2\,(\text{ð}'\,\Phi_{11}\ ) +3\,(\text{ð}'\,\Psi_2\ ) ,$

$\text{þ}\,\Phi_{22}\ \to\, 2\,\Phi_{02}\ \lambda\ -\Phi_{20}\ \overline{\lambda}\ -2\,\Phi_{11}\ \mu\ +9\,\Psi_2\ \mu\ +4\,\Phi_{11}\ \overline{\mu}\ -2\,\Phi_{01}\ \nu\ -6\,\Psi_1\ \nu\ -$

$2\,\Phi_{10}\ \overline{\nu}\ +2\,\Phi_{12}\ \pi\ +2\,\Phi_{21}\ \overline{\pi}\ -2\,\Phi_{22}\ \rho\ +\Phi_{22}\ \overline{\rho}\ -3\,\Psi_4\ \sigma\ +2\,\Phi_{21}\ \tau\ +6\,\Psi_3\ \tau\ +$

$2\,\Phi_{12}\ \overline{\tau}\ +2\,(\text{þ}'\,\Phi_{11}\ ) +3\,(\text{þ}'\,\Psi_2\ ) +\text{ð}\,\Phi_{21}\ -3\,(\text{ð}\,\Psi_3\ ) -2\,(\text{ð}'\,\Phi_{12}\ ),\ \text{þ}\,\Psi_4\ \to$

$2\,\Phi_{11}\ \lambda\ -3\,\Psi_2\ \lambda\ +\Phi_{20}\ \overline{\mu}\ -2\,\Phi_{10}\ \nu\ +4\,\Psi_3\ \pi\ +\Psi_4\ \rho\ -\Phi_{22}\ \overline{\sigma}\ +2\,\Phi_{21}\ \overline{\tau}\ +\text{þ}'\,\Phi_{20}\ -\text{ð}'\,\Phi_{21}\ +\text{ð}'\,\Psi_3\ ,$

$\text{þ}'\,\Lambda\ \to\, -\Phi_{02}\ \lambda\ -3\,\Psi_2\ \mu\ -2\,\Phi_{11}\ \overline{\mu}\ +\Phi_{01}\ \nu\ +2\,\Psi_1\ \nu\ +\Phi_{10}\ \overline{\nu}\ +\Phi_{22}\ \rho\ +$

$\Psi_4\ \sigma\ -\Phi_{21}\ \tau\ -2\,\Psi_3\ \tau\ -\Phi_{12}\ \overline{\tau}\ -\text{þ}'\,\Phi_{11}\ -\text{þ}'\,\Psi_2\ +\text{ð}\,\Psi_3\ +\text{ð}'\,\Phi_{12}\ ,$

$\text{þ}'\,\Phi_{01}\ \to\, -2\,\Psi_1\ \mu\ -2\,\Phi_{01}\ \overline{\mu}\ +\Psi_0\ \nu\ +\Phi_{00}\ \overline{\nu}\ +2\,\Phi_{12}\ \rho\ +2\,\Psi_3\ \sigma\ -2\,\Phi_{11}\ \tau\ -$

$3\,\Psi_2\ \tau\ -\Phi_{02}\ \overline{\tau}\ -\text{þ}'\,\Psi_1\ -2\,(\text{ð}\,\Lambda\ ) +\text{ð}\,\Psi_2\ +\text{ð}'\,\Phi_{02}\ ,\ \text{þ}'\,\Phi_{21}\ \to$

$-2\,\Phi_{12}\ \lambda\ -4\,\Psi_3\ \mu\ -2\,\Phi_{21}\ \overline{\mu}\ +2\,\Phi_{11}\ \nu\ +3\,\Psi_2\ \nu\ +\Phi_{20}\ \overline{\nu}\ -\Psi_4\ \tau\ -\Phi_{22}\ \overline{\tau}\ -\text{þ}'\,\Psi_3\ +\text{ð}\,\Psi_4\ +\text{ð}'\,\Phi_{22}\ \Big\}$

In[361]:= **FullSimplify //@ %**

Out[361]= $\{ \text{þ} \Lambda \rightarrow \frac{1}{2} (2 \Psi_3 \kappa + \Psi_0 \lambda - \Phi_{00} \overline{\mu} - 2 \Psi_1 \pi + 2 \Phi_{11} \rho - $
$3 \Psi_2 \rho + \Phi_{02} \overline{\sigma} - 2 \Phi_{10} \tau - 2 \Phi_{01} \overline{\tau} + \text{þ} \Psi_2 - \text{þ}' \Phi_{00} + \eth \Phi_{01} - \eth' \Psi_1), \text{þ} \Phi_{01} \rightarrow$
$-2 \Phi_{11} \kappa - 3 \Psi_2 \kappa - \Phi_{02} \overline{\kappa} + \Psi_0 \pi + \Phi_{00} \overline{\pi} + 4 \Psi_1 \rho + 2 \Phi_{01} \overline{\rho} + 2 \Phi_{10} \sigma - \text{þ} \Psi_1 + \eth \Phi_{00} + \eth' \Psi_0 ,$

$\text{þ} \Phi_{02} \rightarrow -2 \Phi_{12} \kappa - \Phi_{00} \overline{\lambda} + \Psi_0 \mu + 2 \Phi_{01} \overline{\pi} + \Phi_{02} \overline{\rho} +$
$2 \Phi_{11} \sigma - 3 \Psi_2 \sigma + 4 \Psi_1 \tau + \text{þ}' \Psi_0 + \eth \Phi_{01} - \eth \Psi_1 ,$

$\text{þ} \Phi_{11} \rightarrow \frac{1}{2} (-2 (\Phi_{21} + 3 \Psi_3) \kappa - 2 \Phi_{12} \overline{\kappa} - 3 \Psi_0 \lambda + \Phi_{00} (-2 \mu + \overline{\mu}) + 6 \Psi_1 \pi -$
$2 \Phi_{11} \rho + 9 \Psi_2 \rho + 4 \Phi_{11} \overline{\rho} + 2 \Phi_{20} \sigma - \Phi_{02} \overline{\sigma} + 2 \Phi_{10} (\overline{\pi} + \tau) +$
$2 \Phi_{01} (\pi + \overline{\tau}) - 3 (\text{þ} \Psi_2) + \text{þ}' \Phi_{00} + 2 (\eth \Phi_{10}) - \eth' \Phi_{01} + 3 (\eth' \Psi_1)),$

$\text{þ} \Phi_{12} \rightarrow -\Phi_{22} \kappa - \Phi_{10} \overline{\lambda} - \Phi_{01} \mu + 2 \Psi_1 \mu - \Psi_0 \nu + \Phi_{02} \pi + 2 \Phi_{11} \overline{\pi} + \Phi_{12} \overline{\rho} +$
$\Phi_{21} \sigma - 2 \Psi_3 \sigma + 3 \Psi_2 \tau + \text{þ}' \Psi_1 - \eth \Lambda + \eth \Phi_{11} - \eth \Psi_2 ,$

$\text{þ} \Phi_{21} \rightarrow -3 \Psi_4 \kappa - \Phi_{22} \overline{\kappa} + 2 \Phi_{01} \lambda - 6 \Psi_1 \lambda - 2 \Phi_{10} \mu + 2 \Phi_{10} \overline{\mu} - 2 \Phi_{00} \nu +$
$2 \Phi_{11} \pi + 9 \Psi_2 \pi + \Phi_{20} \overline{\pi} - 2 \Phi_{21} \rho + 6 \Psi_3 \rho + 2 \Phi_{21} \overline{\rho} - 2 \Phi_{12} \overline{\sigma} +$
$2 \Phi_{20} \tau + 4 \Phi_{11} \overline{\tau} - 3 (\text{þ} \Psi_3) + 2 (\text{þ}' \Phi_{10}) + \eth \Phi_{20} - 2 (\eth \Phi_{11}) + 3 (\eth' \Psi_2),$

$\text{þ} \Phi_{22} \rightarrow 2 \Phi_{02} \lambda - \Phi_{20} \overline{\lambda} - 2 \Phi_{11} \mu + 9 \Psi_2 \mu + 4 \Phi_{11} \overline{\mu} - 2 (\Phi_{01} + 3 \Psi_1) \nu -$
$2 \Phi_{10} \overline{\nu} - 2 \Phi_{22} \rho + \Phi_{22} \overline{\rho} - 3 \Psi_4 \sigma + 6 \Psi_3 \tau + 2 \Phi_{21} (\overline{\pi} + \tau) +$
$2 \Phi_{12} (\pi + \overline{\tau}) + 2 (\text{þ}' \Phi_{11}) + 3 (\text{þ}' \Psi_2) + \eth \Phi_{21} - 3 (\eth \Psi_3) - 2 (\eth' \Phi_{12}), \text{þ} \Psi_4 \rightarrow$
$2 \Phi_{11} \lambda - 3 \Psi_2 \lambda + \Phi_{20} \overline{\mu} - 2 \Phi_{10} \nu + 4 \Psi_3 \pi + \Psi_4 \rho - \Phi_{22} \overline{\sigma} + 2 \Phi_{21} \overline{\tau} + \text{þ}' \Phi_{20} - \eth' \Phi_{21} + \eth' \Psi_3 ,$

$\text{þ}' \Lambda \rightarrow -\Phi_{02} \lambda - 3 \Psi_2 \mu - 2 \Phi_{11} \overline{\mu} + (\Phi_{01} + 2 \Psi_1) \nu + \Phi_{10} \overline{\nu} + \Phi_{22} \rho +$
$\Psi_4 \sigma - (\Phi_{21} + 2 \Psi_3) \tau - \Phi_{12} \overline{\tau} - \text{þ}' \Phi_{11} - \text{þ}' \Psi_2 + \eth \Psi_3 + \eth' \Phi_{12} ,$

$\text{þ}' \Phi_{01} \rightarrow -2 \Psi_1 \mu - 2 \Phi_{01} \overline{\mu} + \Psi_0 \nu + \Phi_{00} \overline{\nu} + 2 \Phi_{12} \rho + 2 \Psi_3 \sigma - 2 \Phi_{11} \tau -$
$3 \Psi_2 \tau - \Phi_{02} \overline{\tau} - \text{þ}' \Psi_1 - 2 (\eth \Lambda) + \eth \Psi_2 + \eth' \Phi_{02} , \text{þ}' \Phi_{21} \rightarrow$
$-2 \Phi_{12} \lambda - 4 \Psi_3 \mu - 2 \Phi_{21} \overline{\mu} + 2 \Phi_{11} \nu + 3 \Psi_2 \nu + \Phi_{20} \overline{\nu} - \Psi_4 \tau - \Phi_{22} \overline{\tau} - \text{þ}' \Psi_3 + \eth \Psi_4 + \eth' \Phi_{22} \}$

In[362]:= **% /. Rule -> Equal**

Out[362]= $\{ \text{þ} \Lambda == \frac{1}{2} (2 \Psi_3 \kappa + \Psi_0 \lambda - \Phi_{00} \overline{\mu} - 2 \Psi_1 \pi + 2 \Phi_{11} \rho - $
$3 \Psi_2 \rho + \Phi_{02} \overline{\sigma} - 2 \Phi_{10} \tau - 2 \Phi_{01} \overline{\tau} + \text{þ} \Psi_2 - \text{þ}' \Phi_{00} + \eth \Phi_{01} - \eth' \Psi_1), \text{þ} \Phi_{01} ==$
$-2 \Phi_{11} \kappa - 3 \Psi_2 \kappa - \Phi_{02} \overline{\kappa} + \Psi_0 \pi + \Phi_{00} \overline{\pi} + 4 \Psi_1 \rho + 2 \Phi_{01} \overline{\rho} + 2 \Phi_{10} \sigma - \text{þ} \Psi_1 + \eth \Phi_{00} + \eth' \Psi_0 ,$

$\text{þ} \Phi_{02} == -2 \Phi_{12} \kappa - \Phi_{00} \overline{\lambda} + \Psi_0 \mu + 2 \Phi_{01} \overline{\pi} + \Phi_{02} \overline{\rho} +$
$2 \Phi_{11} \sigma - 3 \Psi_2 \sigma + 4 \Psi_1 \tau + \text{þ}' \Psi_0 + \eth \Phi_{01} - \eth \Psi_1 ,$

$\text{þ} \Phi_{11} == \frac{1}{2} (-2 (\Phi_{21} + 3 \Psi_3) \kappa - 2 \Phi_{12} \overline{\kappa} - 3 \Psi_0 \lambda + \Phi_{00} (-2 \mu + \overline{\mu}) + 6 \Psi_1 \pi -$
$2 \Phi_{11} \rho + 9 \Psi_2 \rho + 4 \Phi_{11} \overline{\rho} + 2 \Phi_{20} \sigma - \Phi_{02} \overline{\sigma} + 2 \Phi_{10} (\overline{\pi} + \tau) +$
$2 \Phi_{01} (\pi + \overline{\tau}) - 3 (\text{þ} \Psi_2) + \text{þ}' \Phi_{00} + 2 (\eth \Phi_{10}) - \eth' \Phi_{01} + 3 (\eth' \Psi_1)),$

$\text{þ} \Phi_{12} == -\Phi_{22} \kappa - \Phi_{10} \overline{\lambda} - \Phi_{01} \mu + 2 \Psi_1 \mu - \Psi_0 \nu + \Phi_{02} \pi + 2 \Phi_{11} \overline{\pi} + \Phi_{12} \overline{\rho} +$
$\Phi_{21} \sigma - 2 \Psi_3 \sigma + 3 \Psi_2 \tau + \text{þ}' \Psi_1 - \eth \Lambda + \eth \Phi_{11} - \eth \Psi_2 ,$

$\text{þ} \Phi_{21} == -3 \Psi_4 \kappa - \Phi_{22} \overline{\kappa} + 2 \Phi_{01} \lambda - 6 \Psi_1 \lambda - 2 \Phi_{10} \mu + 2 \Phi_{10} \overline{\mu} - 2 \Phi_{00} \nu +$
$2 \Phi_{11} \pi + 9 \Psi_2 \pi + \Phi_{20} \overline{\pi} - 2 \Phi_{21} \rho + 6 \Psi_3 \rho + 2 \Phi_{21} \overline{\rho} - 2 \Phi_{12} \overline{\sigma} +$
$2 \Phi_{20} \tau + 4 \Phi_{11} \overline{\tau} - 3 (\text{þ} \Psi_3) + 2 (\text{þ}' \Phi_{10}) + \eth \Phi_{20} - 2 (\eth \Phi_{11}) + 3 (\eth' \Psi_2),$

$\text{þ} \Phi_{22} == 2 \Phi_{02} \lambda - \Phi_{20} \overline{\lambda} - 2 \Phi_{11} \mu + 9 \Psi_2 \mu + 4 \Phi_{11} \overline{\mu} - 2 (\Phi_{01} + 3 \Psi_1) \nu -$
$2 \Phi_{10} \overline{\nu} - 2 \Phi_{22} \rho + \Phi_{22} \overline{\rho} - 3 \Psi_4 \sigma + 6 \Psi_3 \tau + 2 \Phi_{21} (\overline{\pi} + \tau) +$
$2 \Phi_{12} (\pi + \overline{\tau}) + 2 (\text{þ}' \Phi_{11}) + 3 (\text{þ}' \Psi_2) + \eth \Phi_{21} - 3 (\eth \Psi_3) - 2 (\eth' \Phi_{12}), \text{þ} \Psi_4 ==$
$2 \Phi_{11} \lambda - 3 \Psi_2 \lambda + \Phi_{20} \overline{\mu} - 2 \Phi_{10} \nu + 4 \Psi_3 \pi + \Psi_4 \rho - \Phi_{22} \overline{\sigma} + 2 \Phi_{21} \overline{\tau} + \text{þ}' \Phi_{20} - \eth' \Phi_{21} + \eth' \Psi_3 ,$

$\text{þ}' \Lambda == -\Phi_{02} \lambda - 3 \Psi_2 \mu - 2 \Phi_{11} \overline{\mu} + (\Phi_{01} + 2 \Psi_1) \nu + \Phi_{10} \overline{\nu} + \Phi_{22} \rho +$
$\Psi_4 \sigma - (\Phi_{21} + 2 \Psi_3) \tau - \Phi_{12} \overline{\tau} - \text{þ}' \Phi_{11} - \text{þ}' \Psi_2 + \eth \Psi_3 + \eth' \Phi_{12} ,$

$\text{þ}' \Phi_{01} == -2 \Psi_1 \mu - 2 \Phi_{01} \overline{\mu} + \Psi_0 \nu + \Phi_{00} \overline{\nu} + 2 \Phi_{12} \rho + 2 \Psi_3 \sigma - 2 \Phi_{11} \tau -$
$3 \Psi_2 \tau - \Phi_{02} \overline{\tau} - \text{þ}' \Psi_1 - 2 (\eth \Lambda) + \eth \Psi_2 + \eth' \Phi_{02} , \text{þ}' \Phi_{21} ==$
$-2 \Phi_{12} \lambda - 4 \Psi_3 \mu - 2 \Phi_{21} \overline{\mu} + 2 \Phi_{11} \nu + 3 \Psi_2 \nu + \Phi_{20} \overline{\nu} - \Psi_4 \tau - \Phi_{22} \overline{\tau} - \text{þ}' \Psi_3 + \eth \Psi_4 + \eth' \Phi_{22} \}$

Previous set of equations is known as GHP Bianchi identities or compacted Bianchi Identities.

In[363]:= **Length[GHPBianchi = %]**

Out[363]= 11

■ 4. Quick start

In this section we show how to use in practical applications the NP and GHP equations computed in previous sections. This is intended to be a short tutorial which enables the user to start working as quick as possible with the equations of the NP and GHP formalisms, so one does not need to read the contents of previous sections to use it. However, to take full advantage of the flexibility offered by our implementation and the remaining features of xAct`, the complete knowledge of sections 1,2 and 3 is needed

The basic variables of the NP and GHP formalisms are the spin coefficients and the components of the curvature tensor. These are stored in lists.

In[364]:= **NPSpinCoefficients**

Out[364]= $\{\alpha, \beta, \gamma, \epsilon, \kappa, \lambda, \mu, \nu, \pi, \rho, \sigma, \tau\}$

In[365]:= **NPCVComponents**

Out[365]= $\{\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Phi_{00}, \Phi_{01}, \Phi_{02}, \Phi_{10}, \Phi_{11}, \Phi_{12}, \Phi_{20}, \Phi_{21}, \Phi_{22}, \Lambda\}$

To type any of these quantities you need to know the imput form of them. This is achieved with the *Mathematica* command InputForm. For example, let us display the input form of the spin coefficient γ :

In[366]:= **NPSpinCoefficients[[3]] // InputForm**

Out[366]//InputForm=
γDyad[ ]

Recall that Greek letters are typed by using the key "esc" followed by the Latin name of the letter followed by "esc". Hence

In[367]:= **γ Dyad[]**

Out[367]= γ

Other essential ingredient of the NP and GHP formalisms are the *frame differentiations.* In the NP formalism these are just the directional derivatives associated to the four elements of the null tetrad. To see how they are typed, we consider their action on the spin coefficient γ .

In[368]:= **{PDNP[{1, -NP}]@γ Dyad[], PDNP[{2, -NP}]@γ Dyad[],**
    PDNP[{3, -NP}]@γ Dyad[], PDNP[{4, -NP}]@γ Dyad[]}

Out[368]= $\{D\gamma, \triangle\gamma, \delta\gamma, \overline{\delta}\gamma\}$

The head PDNP denotes a generic NP operator associated to the null tetrad, whereas the arguments {1,-NP}, ... {4,-NP} are used to refer to each of the directional derivatives of the null tetrad elements. In this way {1,-NP} means the first element of the null tetrad, {2,-NP} the second element and so on. The elements of the null tetrad are numbered according to the usual convention. In the case of the GHP formalism, we have to take into account that only weighted quantities appear in the formalism. All the components of the Riemann tensor are weighted, but not all the spin coefficients are so. The weighted spin coefficients are stored in the following list.

In[369]:= **GHPSpinCoefficients**

Out[369]= $\{\kappa, \lambda, \mu, \nu, \pi, \rho, \sigma, \tau\}$

Let us take κ : the action of the four GHP operators on it is

In[370]:= **{Θ[{1, -NP}]@κ Dyad[], Θ[{2, -NP}]@κ Dyad[], Θ[{3, -NP}]@κ Dyad[], Θ[{4, -NP}]@κ Dyad[]}**

Out[370]= $\{$þκ , þ'κ , ðκ , ð'κ $\}$

In this case, the head $\ominus$ (typed esc-Q-esc) is the notation for a generic GHP differential operator while

$\{1,-NP\},...,\{4,-NP\}$ are, as before, the labels for each of the elements of the null tetrad. We can transform the GHP differentials into NP differentials as follows.

In[371]:= **% /. GHPToNP**

Out[371]= $\left\{ (-3\,\epsilon \ -\overline{\epsilon}\ )\ \kappa\ + D\,\kappa\ ,\ (-3\,\gamma\ -\overline{\gamma}\ )\ \kappa\ +\Delta\,\kappa\ ,\ (-\overline{\alpha}\ -3\,\beta\ )\ \kappa\ +\delta\,\kappa\ ,\ \left(-3\,\alpha\ -\overline{\beta}\ \right)\ \kappa\ +\overline{\delta}\,\kappa\ \right\}$

NP differentials can be also transformed back into GHP differentials.

In[372]:= **% /. NPToGHP**

Out[372]= $\left\{ 3\,\epsilon\ \kappa\ + (-3\,\epsilon\ -\overline{\epsilon}\ )\ \kappa\ +\overline{\epsilon}\ \kappa\ +$þ$\kappa\ ,\ 3\,\gamma\ \kappa\ + (-3\,\gamma\ -\overline{\gamma}\ )\ \kappa\ +\overline{\gamma}\ \kappa\ +$þ'$\kappa\ ,$

$\overline{\alpha}\ \kappa\ + (-\overline{\alpha}\ -3\,\beta\ )\ \kappa\ +3\,\beta\ \kappa\ +$ð$\kappa\ ,\ 3\,\alpha\ \kappa\ + \left(-3\,\alpha\ -\overline{\beta}\ \right)\ \kappa\ +\overline{\beta}\ \kappa\ +$ð'$\kappa\ \}$

In[373]:= **% // Simplify**

Out[373]= $\{$þκ , þ'κ , ðκ , ð'κ $\}$

Previous operations can be performed on any weighted quantity and the appropriate weights will be automatically inserted. For example:

In[374]:= **{Θ[{1, -NP}]@PsiCDe4Dyad[], Θ[{2, -NP}]@PsiCDe4Dyad[],**
** Θ[{3, -NP}]@PsiCDe4Dyad[], Θ[{4, -NP}]@PsiCDe4Dyad[]}**

Out[374]= $\{$þΨ_4 , þ'Ψ_4 , ðΨ_4 , ð'Ψ_4 $\}$

In[375]:= **% //. GHPToNP**

Out[375]= $\left\{ 4\,\Psi_4\ \epsilon\ + D\,\Psi_4\ ,\ 4\,\Psi_4\ \gamma\ +\Delta\,\Psi_4\ ,\ 4\,\Psi_4\ \beta\ +\delta\,\Psi_4\ ,\ 4\,\Psi_4\ \alpha\ +\overline{\delta}\,\Psi_4\ \right\}$

The basic equations of each formalism are stored in lists. These are the names of the lists and the correspondences:

Newman-Penrose commutation relations:

In[376]:= **NPCommutators**

Out[376]= $\Big\{ D\,\Delta\,\psi\ == -\,(\gamma\ +\overline{\gamma}\ )\ (D\,\psi\ ) - (\epsilon\ +\overline{\epsilon}\ )\ (\Delta\,\psi\ ) +\Delta\,D\,\psi\ + (\pi\ +\overline{\tau}\ )\ (\delta\,\psi\ ) + (\overline{\pi}\ +\tau\ )\ \left(\overline{\delta}\,\psi\ \right),$

$D\,\delta\,\psi\ == -\,(\overline{\alpha}\ +\beta\ -\overline{\pi}\ )\ (D\,\psi\ ) -\kappa\ (\Delta\,\psi\ ) + (\epsilon\ -\overline{\epsilon}\ +\overline{\rho}\ )\ (\delta\,\psi\ ) +\delta\,D\,\psi\ +\sigma\ \left(\overline{\delta}\,\psi\ \right),$

$\Delta\,\delta\,\psi\ == \overline{\nu}\ (D\,\psi\ ) + (\overline{\alpha}\ +\beta\ -\tau\ )\ (\Delta\,\psi\ ) + (\gamma\ -\overline{\gamma}\ -\mu\ )\ (\delta\,\psi\ ) +\delta\,\Delta\,\psi\ -\overline{\lambda}\ \left(\overline{\delta}\,\psi\ \right),$

$\delta\,\overline{\delta}\,\psi\ == (\mu\ -\overline{\mu}\ )\ (D\,\psi\ ) + (\rho\ -\overline{\rho}\ )\ (\Delta\,\psi\ ) + \left(-\alpha\ +\overline{\beta}\ \right)\ (\delta\,\psi\ ) + (\overline{\alpha}\ -\beta\ )\ \left(\overline{\delta}\,\psi\ \right) +\overline{\delta}\,\delta\,\psi\ \Big\}$

Newman-Penrose Ricci identities:

In[377]:= **NPRicci**

Out[377]= $\{ D\,\alpha \;==\; -\Phi_{10}\; -2\,\alpha\;\epsilon\; -\overline{\beta}\;\epsilon\; +\alpha\;\overline{\epsilon}\; -\gamma\;\overline{\kappa}\; -\kappa\;\lambda\; +\epsilon\;\pi\; +\alpha\;\rho\; +\pi\;\rho\; +\beta\;\overline{\sigma}\; +\overline{\delta}\,\epsilon\;,$

$\quad D\,\beta \;==\; \Psi_1\; -\overline{\alpha}\;\epsilon\; -\beta\;\overline{\epsilon}\; -\gamma\;\kappa\; -\kappa\;\mu\; +\epsilon\;\overline{\pi}\; +\beta\;\overline{\rho}\; +\alpha\;\sigma\; +\pi\;\sigma\; +\delta\,\epsilon\;,$

$\quad D\,\gamma \;==\; \Lambda\; -\Phi_{11}\; +\Psi_2\; -2\,\gamma\;\epsilon\; -\overline{\gamma}\;\epsilon\; -\gamma\;\overline{\epsilon}\; -\kappa\;\nu\; +\beta\;\pi\; +\alpha\;\overline{\pi}\; +\alpha\;\tau\; +\pi\;\tau\; +\beta\;\overline{\tau}\; +\Delta\,\epsilon\;,$

$\quad D\,\lambda \;==\; -\Phi_{20}\; -3\,\epsilon\;\lambda\; +\overline{\epsilon}\;\lambda\; -\overline{\kappa}\;\nu\; +\alpha\;\pi\; -\overline{\beta}\;\pi\; +\pi^{\,2}\; +\lambda\;\rho\; +\mu\;\overline{\sigma}\; +\overline{\delta}\,\pi\;,$

$\quad D\,\mu \;==\; -2\,\Lambda\; +\Psi_2\; -\epsilon\;\mu\; -\overline{\epsilon}\;\mu\; -\kappa\;\nu\; -\overline{\alpha}\;\pi\; +\beta\;\pi\; +\pi\;\overline{\pi}\; +\mu\;\overline{\rho}\; +\lambda\;\sigma\; +\delta\,\pi\;,$

$\quad D\,\nu \;==\; -\Phi_{21}\; +\Psi_3\; -3\,\epsilon\;\nu\; -\overline{\epsilon}\;\nu\; +\gamma\;\pi\; -\overline{\gamma}\;\pi\; +\mu\;\pi\; +\lambda\;\overline{\pi}\; +\lambda\;\tau\; +\mu\;\overline{\tau}\; +\Delta\,\pi\;,$

$\quad D\,\rho \;==\; -\Phi_{00}\; -3\,\alpha\;\kappa\; -\overline{\beta}\;\kappa\; +\kappa\;\pi\; +\epsilon\;\rho\; +\overline{\epsilon}\;\rho\; +\rho^{\,2}\; +\sigma\;\overline{\sigma}\; -\overline{\kappa}\;\tau\; +\overline{\delta}\,\kappa\;,$

$\quad D\,\sigma \;==\; \Psi_0\; -\overline{\alpha}\;\kappa\; -3\,\beta\;\kappa\; +\kappa\;\overline{\pi}\; +3\,\epsilon\;\sigma\; -\overline{\epsilon}\;\sigma\; +\rho\;\sigma\; +\overline{\rho}\;\sigma\; -\kappa\;\tau\; +\delta\,\kappa\;,$

$\quad D\,\tau \;==\; -\Phi_{01}\; +\Psi_1\; -3\,\gamma\;\kappa\; -\overline{\gamma}\;\kappa\; +\overline{\pi}\;\rho\; +\pi\;\sigma\; +\epsilon\;\tau\; -\overline{\epsilon}\;\tau\; +\rho\;\tau\; +\sigma\;\overline{\tau}\; +\Delta\,\kappa\;,$

$\quad \Delta\,\alpha \;==\; -\Psi_3\; +\overline{\beta}\;\gamma\; +\alpha\;\overline{\gamma}\; -\beta\;\lambda\; -\alpha\;\overline{\mu}\; +\epsilon\;\nu\; +\nu\;\rho\; -\lambda\;\tau\; -\gamma\;\overline{\tau}\; +\overline{\delta}\,\gamma\;,$

$\quad \Delta\,\beta \;==\; \Phi_{12}\; +\overline{\alpha}\;\gamma\; +2\,\beta\;\gamma\; -\beta\;\overline{\gamma}\; -\alpha\;\overline{\lambda}\; -\beta\;\mu\; +\epsilon\;\overline{\nu}\; +\nu\;\sigma\; -\gamma\;\tau\; -\mu\;\tau\; +\delta\,\gamma\;,$

$\quad \Delta\,\lambda \;==\; -\Psi_4\; -3\,\gamma\;\lambda\; +\overline{\gamma}\;\lambda\; -\lambda\;\mu\; -\lambda\;\overline{\mu}\; +3\,\alpha\;\nu\; +\overline{\beta}\;\nu\; +\nu\;\pi\; -\nu\;\overline{\tau}\; +\overline{\delta}\,\nu\;,$

$\quad \Delta\,\mu \;==\; \Phi_{22}\; -\lambda\;\overline{\lambda}\; -\gamma\;\mu\; -\overline{\gamma}\;\mu\; -\mu^{\,2}\; +\overline{\alpha}\;\nu\; +3\,\beta\;\nu\; +\overline{\nu}\;\pi\; -\nu\;\tau\; +\delta\,\nu\;,$

$\quad \Delta\,\rho \;==\; 2\,\Lambda\; -\Psi_2\; +\kappa\;\nu\; +\gamma\;\rho\; +\overline{\gamma}\;\rho\; -\overline{\mu}\;\rho\; -\lambda\;\sigma\; -\alpha\;\tau\; +\overline{\beta}\;\tau\; -\tau\;\overline{\tau}\; +\overline{\delta}\,\tau\;,$

$\quad \Delta\,\sigma \;==\; \Phi_{02}\; +\kappa\;\overline{\nu}\; -\overline{\lambda}\;\rho\; +3\,\gamma\;\sigma\; -\overline{\gamma}\;\sigma\; -\mu\;\sigma\; +\overline{\alpha}\;\tau\; -\beta\;\tau\; -\tau^{\,2}\; +\delta\,\tau\;,$

$\quad \delta\,\alpha \;==\; -\Lambda\; -\Phi_{11}\; -\Psi_2\; +\alpha\;\overline{\alpha}\; -2\,\alpha\;\beta\; +\beta\;\overline{\beta}\; +\epsilon\;\mu\; -\epsilon\;\overline{\mu}\; +\gamma\;\rho\; +\mu\;\rho\; -\gamma\;\overline{\rho}\; -\lambda\;\sigma\; +\overline{\delta}\,\beta\;,$

$\quad \delta\,\lambda \;==\; -\Phi_{21}\; -\Psi_3\; +\overline{\alpha}\;\lambda\; -3\,\beta\;\lambda\; +\alpha\;\mu\; +\overline{\beta}\;\mu\; +\mu\;\pi\; -\overline{\mu}\;\pi\; +\nu\;\rho\; -\nu\;\overline{\rho}\; +\overline{\delta}\,\mu\;,$

$\quad \delta\,\rho \;==\; -\Phi_{01}\; -\Psi_1\; +\kappa\;\mu\; -\kappa\;\overline{\mu}\; +\overline{\alpha}\;\rho\; +\beta\;\rho\; -3\,\alpha\;\sigma\; +\overline{\beta}\;\sigma\; +\rho\;\tau\; -\overline{\rho}\;\tau\; +\overline{\delta}\,\sigma\,\}$

Newman-Penrose Bianchi identities:

In[378]:= **NPBianchi**

Out[378]= $\{ D\,\Phi_{01} \;==\; 2\,\Psi_1\;\epsilon\; -2\,\Phi_{11}\;\kappa\; -3\,\Psi_2\;\kappa\; -\Phi_{02}\;\overline{\kappa}\; +\Psi_0\;(-4\,\alpha\; +\pi\;)\; +$

$\quad \Phi_{00}\;(-2\,(\overline{\alpha}\; +\beta\;)\; +\overline{\pi}\;)\; +4\,\Psi_1\;\rho\; +2\,\Phi_{01}\;(\epsilon\; +\overline{\rho}\;)\; +2\,\Phi_{10}\;\sigma\; -D\,\Psi_1\; +\delta\,\Phi_{00}\; +\overline{\delta}\,\Psi_0\;,$

$\quad D\,\Phi_{02} \;==\; -2\,\Phi_{01}\;\beta\; +2\,\Psi_1\;\beta\; -2\,\Phi_{12}\;\kappa\; -\Phi_{00}\;\overline{\lambda}\; +\Psi_0\;(-4\,\gamma\; +\mu\;)\; +2\,\Phi_{01}\;\overline{\pi}\; +$

$\quad \Phi_{02}\;(2\,\epsilon\; -2\,\overline{\epsilon}\; +\overline{\rho}\;)\; +2\,\Phi_{11}\;\sigma\; -3\,\Psi_2\;\sigma\; +4\,\Psi_1\;\tau\; +\Delta\,\Psi_0\; +\delta\,\Phi_{01}\; -\delta\,\Psi_1\;,$

$\quad D\,\Phi_{11} \;==\; -(\Phi_{21}\; +2\,\Psi_3\;)\;\kappa\; -\Phi_{12}\;\overline{\kappa}\; -\Psi_0\;\lambda\; -\Phi_{00}\;\mu\; +\Phi_{01}\;\pi\; +2\,\Psi_1\;(-\alpha\; +\pi\;)\; +$

$\quad \Phi_{10}\;(-2\,\overline{\alpha}\; +\overline{\pi}\;)\; +3\,\Psi_2\;\rho\; +2\,\Phi_{11}\;\overline{\rho}\; +\Phi_{20}\;\sigma\; -D\,\Lambda\; -D\,\Psi_2\; +\delta\,\Phi_{10}\; +\overline{\delta}\,\Psi_1\;,$

$\quad D\,\Phi_{22} \;==\; -\Phi_{20}\;\overline{\lambda}\; -2\,\Phi_{11}\;\mu\; +3\,\Psi_2\;\mu\; -2\,\Psi_1\;\nu\; +2\,\Phi_{12}\;\pi\; +2\,\Phi_{21}\;(\beta\; +\overline{\pi}\;)\; +$

$\quad \Phi_{22}\;(-2\,(\epsilon\; +\overline{\epsilon}\;)\; +\overline{\rho}\;)\; -\Psi_4\;\sigma\; +2\,\Psi_3\;(-\beta\; +\tau\;)\; -2\,(\Delta\,\Lambda\;)\; +\Delta\,\Psi_2\; +\delta\,\Phi_{21}\; -\delta\,\Psi_3\;,$

$\quad D\,\Psi_4 \;==\; -2\,\Phi_{21}\;\alpha\; +2\,\Psi_3\;\alpha\; -4\,\Psi_4\;\epsilon\; +2\,\Phi_{11}\;\lambda\; -3\,\Psi_2\;\lambda\; +\Phi_{20}\;(2\,\gamma\; -2\,\overline{\gamma}\; +\overline{\mu}\;)\; -$

$\quad 2\,\Phi_{10}\;\nu\; +4\,\Psi_3\;\pi\; +\Psi_4\;\rho\; -\Phi_{22}\;\overline{\sigma}\; +2\,\Phi_{21}\;\overline{\tau}\; +\Delta\,\Phi_{20}\; -\overline{\delta}\,\Phi_{21}\; +\overline{\delta}\,\Psi_3\;,$

$\quad \Delta\,\Phi_{00} \;==\; 2\,\Phi_{00}\;(\gamma\; +\overline{\gamma}\;)\; +2\,\Psi_3\;\kappa\; +\Psi_0\;\lambda\; -\Phi_{00}\;\overline{\mu}\; +2\,\Psi_1\;(\alpha\; -\pi\;)\; +2\,\Phi_{11}\;\rho\; -$

$\quad 3\,\Psi_2\;\rho\; +\Phi_{02}\;\overline{\sigma}\; -2\,\Phi_{10}\;\tau\; -2\,\Phi_{01}\;(\alpha\; +\overline{\tau}\;)\; -2\,(D\,\Lambda\;)\; +D\,\Psi_2\; +\overline{\delta}\,\Phi_{01}\; -\overline{\delta}\,\Psi_1\;,$

$\quad \Delta\,\Phi_{11} \;==\; 2\,\Psi_3\;\beta\; +2\,\Phi_{12}\;\overline{\beta}\; -\Phi_{02}\;\lambda\; -3\,\Psi_2\;\mu\; -2\,\Phi_{11}\;\overline{\mu}\; +\Phi_{01}\;\nu\; +2\,\Psi_1\;\nu\; +\Phi_{10}\;\overline{\nu}\; +$

$\quad \Phi_{22}\;\rho\; +\Psi_4\;\sigma\; -\Phi_{21}\;\tau\; -2\,\Psi_3\;\tau\; -\Phi_{12}\;\overline{\tau}\; -\Delta\,\Lambda\; -\Delta\,\Psi_2\; +\delta\,\Psi_3\; +\overline{\delta}\,\Phi_{12}\;,$

$\quad \Delta\,\Phi_{21} \;==\; 2\,\Phi_{22}\;\left(\alpha\; +\overline{\beta}\;\right)\; -2\,\Phi_{12}\;\lambda\; -2\,(\Psi_3\;(\gamma\; +2\,\mu\;)\; +\Phi_{21}\;(\gamma\; +\overline{\mu}\;))\; +2\,\Phi_{11}\;\nu\; +$

$\quad 3\,\Psi_2\;\nu\; +\Phi_{20}\;\overline{\nu}\; +\Psi_4\;(4\,\beta\; -\tau\;)\; -\Phi_{22}\;\overline{\tau}\; -\Delta\,\Psi_3\; +\delta\,\Psi_4\; +\overline{\delta}\,\Phi_{22}\;,$

$\quad \delta\,\Phi_{11} \;==\; 2\,\Psi_1\;\gamma\; +2\,\Phi_{12}\;\overline{\epsilon}\; +\Phi_{22}\;\kappa\; +\Phi_{10}\;\overline{\lambda}\; +\Phi_{01}\;\mu\; -2\,\Psi_1\;\mu\; +\Psi_0\;\nu\; -\Phi_{02}\;\pi\; -$

$\quad 2\,\Phi_{11}\;\overline{\pi}\; -\Phi_{12}\;\overline{\rho}\; -\Phi_{21}\;\sigma\; +2\,\Psi_3\;\sigma\; -3\,\Psi_2\;\tau\; +D\,\Phi_{12}\; -\Delta\,\Psi_1\; +\delta\,\Lambda\; +\delta\,\Psi_2\;,$

$\quad \overline{\delta}\,\Phi_{02} \;==\; -2\,\Phi_{01}\;\gamma\; -2\,\Psi_1\;\gamma\; +2\,\Psi_1\;\mu\; +2\,\Phi_{01}\;\overline{\mu}\; -\Psi_0\;\nu\; -\Phi_{00}\;\overline{\nu}\; -2\,\Phi_{12}\;\rho\; -2\,\Psi_3\;\sigma\; +$

$\quad 2\,\Phi_{11}\;\tau\; +3\,\Psi_2\;\tau\; +\Phi_{02}\;\left(2\,\alpha\; -2\,\overline{\beta}\; +\overline{\tau}\;\right)\; +\Delta\,\Phi_{01}\; +\Delta\,\Psi_1\; +2\,(\delta\,\Lambda\;)\; -\delta\,\Psi_2\;,$

$\quad 2\,\left(\overline{\delta}\,\Phi_{11}\;\right) \;==\; -6\,\Psi_3\;\epsilon\; -3\,\Psi_4\;\kappa\; -\Phi_{22}\;\overline{\kappa}\; +2\,\Phi_{01}\;\lambda\; -6\,\Psi_1\;\lambda\; -2\,\Phi_{10}\;(2\,\overline{\gamma}\; +\mu\; -\overline{\mu}\;)\; -$

$\quad 2\,\Phi_{00}\;\nu\; +2\,\Phi_{11}\;\pi\; +9\,\Psi_2\;\pi\; +6\,\Psi_3\;\rho\; -2\,\Phi_{21}\;(\epsilon\; +\rho\; -\overline{\rho}\;)\; -2\,\Phi_{12}\;\overline{\sigma}\; +$

$\quad \Phi_{20}\;(-2\,\overline{\alpha}\; +2\,\beta\; +\overline{\pi}\; +2\,\tau\;)\; +4\,\Phi_{11}\;\overline{\tau}\; -D\,\Phi_{21}\; -3\,(D\,\Psi_3\;)\; +2\,(\Delta\,\Phi_{10}\;)\; +\delta\,\Phi_{20}\; +3\,\left(\overline{\delta}\,\Psi_2\;\right)\}$

GHP commutators acting on a generic quantity $\eta^{\{w,\,z\}}$ of boost weight w and spin weight z.

In[379]:= **GHPCommutators**

Out[379]= $\{\text{þ}\,\text{þ}'\,\eta^{\{w,\,z\}} - \text{þ}'\,\text{þ}\,\eta^{\{w,\,z\}} == -\eta^{\{w,\,z\}}\,\big(w\,(\Lambda\,-\Phi_{11}\,+\Psi_2\,-\kappa\,\,\nu\,+\pi\,\,\tau\,) + z\,(\Lambda\,-\Phi_{11}\,+\overline{\Psi}_2\,-\overline{\kappa}\,\,\overline{\nu}\,+\overline{\pi}\,\,\overline{\tau}\,)\big) +$

$\qquad (\pi\,+\overline{\tau}\,)\,\big(\text{ð}\,\eta^{\{w,\,z\}}\big) + (\overline{\pi}\,+\tau\,)\,\big(\text{ð}'\,\eta^{\{w,\,z\}}\big),$

$\quad \text{þ}\,\text{ð}\,\eta^{\{w,\,z\}} - \text{ð}\,\text{þ}\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\,\big(z\,(\Phi_{01}\,+\overline{\kappa}\,\,\overline{\lambda}\,-\overline{\pi}\,\,\overline{\rho}\,) - w\,(\Psi_1\,-\kappa\,\,\mu\,+\pi\,\,\sigma\,)\big) +$

$\qquad \overline{\pi}\,\big(\text{þ}\,\eta^{\{w,\,z\}}\big) - \kappa\,\big(\text{þ}'\,\eta^{\{w,\,z\}}\big) + \overline{\rho}\,\big(\text{ð}\,\eta^{\{w,\,z\}}\big) + \sigma\,\big(\text{ð}'\,\eta^{\{w,\,z\}}\big),$

$\quad \text{þ}'\,\text{ð}\,\eta^{\{w,\,z\}} - \text{ð}\,\text{þ}'\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\,\big(-w\,(\Phi_{12}\,+\nu\,\,\sigma\,-\mu\,\,\tau\,) + z\,(\overline{\Psi}_3\,-\overline{\nu}\,\,\overline{\rho}\,+\overline{\lambda}\,\,\overline{\tau}\,)\big) +$

$\qquad \overline{\nu}\,\big(\text{þ}\,\eta^{\{w,\,z\}}\big) - \tau\,\big(\text{þ}'\,\eta^{\{w,\,z\}}\big) - \mu\,\big(\text{ð}\,\eta^{\{w,\,z\}}\big) - \overline{\lambda}\,\big(\text{ð}'\,\eta^{\{w,\,z\}}\big),$

$\quad \text{þ}\,\text{ð}'\,\eta^{\{w,\,z\}} - \text{ð}'\,\text{þ}\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\,\big(w\,(\Phi_{10}\,+\kappa\,\,\lambda\,-\pi\,\,\rho\,) - z\,(\overline{\Psi}_1\,-\overline{\kappa}\,\,\overline{\mu}\,+\overline{\pi}\,\,\overline{\sigma}\,)\big) +$

$\qquad \pi\,\big(\text{þ}\,\eta^{\{w,\,z\}}\big) - \overline{\kappa}\,\big(\text{þ}'\,\eta^{\{w,\,z\}}\big) + \overline{\sigma}\,\big(\text{ð}\,\eta^{\{w,\,z\}}\big) + \rho\,\big(\text{ð}'\,\eta^{\{w,\,z\}}\big),$

$\quad \text{þ}'\,\text{ð}'\,\eta^{\{w,\,z\}} - \text{ð}'\,\text{þ}'\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\,\big(w\,(\Psi_3\,-\nu\,\,\rho\,+\lambda\,\,\tau\,) - z\,(\Phi_{21}\,+\overline{\nu}\,\,\overline{\sigma}\,-\overline{\mu}\,\,\overline{\tau}\,)\big) +$

$\qquad \nu\,\big(\text{þ}\,\eta^{\{w,\,z\}}\big) - \overline{\tau}\,\big(\text{þ}'\,\eta^{\{w,\,z\}}\big) - \lambda\,\big(\text{ð}\,\eta^{\{w,\,z\}}\big) - \overline{\mu}\,\big(\text{ð}'\,\eta^{\{w,\,z\}}\big),$

$\quad \text{ð}\,\text{ð}'\,\eta^{\{w,\,z\}} - \text{ð}'\,\text{ð}\,\eta^{\{w,\,z\}} == \eta^{\{w,\,z\}}\,\big(w\,(\Lambda\,+\Phi_{11}\,+\Psi_2\,-\mu\,\,\rho\,+\lambda\,\,\sigma\,) - z\,(\Lambda\,+\Phi_{11}\,+\overline{\Psi}_2\,-\overline{\mu}\,\,\overline{\rho}\,+\overline{\lambda}\,\,\overline{\sigma}\,)\big) +$

$\qquad (\mu\,-\overline{\mu}\,)\,\big(\text{þ}\,\eta^{\{w,\,z\}}\big) + (\rho\,-\overline{\rho}\,)\,\big(\text{þ}'\,\eta^{\{w,\,z\}}\big)\}$

GHP Ricci identities:

In[380]:= **GHPRicci**

Out[380]= $\{\text{þ}\,\lambda\,== -\Phi_{20}\,-\overline{\kappa}\,\,\nu\,+\pi^2\,+\lambda\,\,\rho\,+\mu\,\,\overline{\sigma}\,+\text{ð}'\,\pi\,,\ \text{þ}\,\mu\,== -2\,\Lambda\,+\Psi_2\,-\kappa\,\,\nu\,+\pi\,\,\overline{\pi}\,+\mu\,\,\overline{\rho}\,+\lambda\,\,\sigma\,+\text{ð}\,\pi\,,$

$\quad \text{þ}\,\nu\,== -\Phi_{21}\,+\Psi_3\,+\lambda\,\,(\overline{\pi}\,+\tau\,) + \mu\,\,(\pi\,+\overline{\tau}\,) + \text{þ}'\,\pi\,,\ \text{þ}\,\rho\,== -\Phi_{00}\,+\kappa\,\,\pi\,+\rho^2\,+\sigma\,\,\overline{\sigma}\,-\overline{\kappa}\,\,\tau\,+\text{ð}'\,\kappa\,,$

$\quad \text{þ}\,\sigma\,== \Psi_0\,+(\rho\,+\overline{\rho}\,)\,\sigma\,+\kappa\,\,(\overline{\pi}\,-\tau\,) + \text{ð}\,\kappa\,,\ \text{þ}\,\tau\,== -\Phi_{01}\,+\Psi_1\,+\rho\,\,(\overline{\pi}\,+\tau\,) + \sigma\,\,(\pi\,+\overline{\tau}\,) + \text{þ}'\,\kappa\,,$

$\quad \text{þ}'\,\lambda\,== -\Psi_4\,-\lambda\,\,(\mu\,+\overline{\mu}\,) + \nu\,\,(\pi\,-\overline{\tau}\,) + \text{ð}'\,\nu\,,\ \text{þ}'\,\mu\,== \Phi_{22}\,-\lambda\,\,\overline{\lambda}\,-\mu^2\,+\overline{\nu}\,\,\pi\,-\nu\,\,\tau\,+\text{ð}\,\nu\,,$

$\quad \text{þ}'\,\rho\,== 2\,\Lambda\,-\Psi_2\,+\kappa\,\,\nu\,-\overline{\mu}\,\,\rho\,-\lambda\,\,\sigma\,-\tau\,\,\overline{\tau}\,+\text{ð}'\,\tau\,,\ \text{þ}'\,\sigma\,== \Phi_{02}\,+\kappa\,\,\overline{\nu}\,-\overline{\lambda}\,\,\rho\,-\mu\,\,\sigma\,-\tau^2\,+\text{ð}\,\tau\,,$

$\quad \text{ð}\,\lambda\,== -\Phi_{21}\,-\Psi_3\,+\mu\,\,\pi\,-\overline{\mu}\,\,\pi\,+\nu\,\,\rho\,-\nu\,\,\overline{\rho}\,+\text{ð}'\,\mu\,,$

$\quad \text{ð}\,\rho\,== -\Phi_{01}\,-\Psi_1\,+\kappa\,\,\mu\,-\kappa\,\,\overline{\mu}\,+\rho\,\,\tau\,-\overline{\rho}\,\,\tau\,+\text{ð}'\,\sigma\,\}$

GHP Bianchi identities:

In[381]:= **GHPBianchi**

Out[381]= $\{ \text{þ} \Lambda \; == \; \frac{1}{2} \; (2 \, \Psi_3 \; \kappa \; + \Psi_0 \; \lambda \; - \Phi_{00} \; \overline{\mu} \; - 2 \, \Psi_1 \; \pi \; + 2 \, \Phi_{11} \; \rho \; -$

$3 \, \Psi_2 \; \rho \; + \Phi_{02} \; \overline{\sigma} \; - 2 \, \Phi_{10} \; \tau \; - 2 \, \Phi_{01} \; \overline{\tau} \; + \text{þ} \, \Psi_2 \; - \text{þ}' \, \Phi_{00} \; + \text{ð}' \, \Phi_{01} \; - \text{ð}' \, \Psi_1 \;), \; \text{þ} \, \Phi_{01} \; ==$

$- 2 \, \Phi_{11} \; \kappa \; - 3 \, \Psi_2 \; \kappa \; - \Phi_{02} \; \overline{\kappa} \; + \Psi_0 \; \pi \; + \Phi_{00} \; \overline{\pi} \; + 4 \, \Psi_1 \; \rho \; + 2 \, \Phi_{01} \; \overline{\rho} \; + 2 \, \Phi_{10} \; \sigma \; - \text{þ} \, \Psi_1 \; + \text{ð} \, \Phi_{00} \; + \text{ð}' \, \Psi_0 \; ,$

$\text{þ} \, \Phi_{02} \; == \; - 2 \, \Phi_{12} \; \kappa \; - \Phi_{00} \; \overline{\lambda} \; + \Psi_0 \; \mu \; + 2 \, \Phi_{01} \; \overline{\pi} \; + \Phi_{02} \; \overline{\rho} \; +$

$2 \, \Phi_{11} \; \sigma \; - 3 \, \Psi_2 \; \sigma \; + 4 \, \Psi_1 \; \tau \; + \text{þ}' \, \Psi_0 \; + \text{ð} \, \Phi_{01} \; - \text{ð} \, \Psi_1 \; ,$

$\text{þ} \, \Phi_{11} \; == \; \frac{1}{2} \; (- 2 \, (\Phi_{21} \; + 3 \, \Psi_3 \;) \; \kappa \; - 2 \, \Phi_{12} \; \overline{\kappa} \; - 3 \, \Psi_0 \; \lambda \; + \Phi_{00} \; (- 2 \, \mu \; + \overline{\mu} \;) \; + 6 \, \Psi_1 \; \pi \; -$

$2 \, \Phi_{11} \; \rho \; + 9 \, \Psi_2 \; \rho \; + 4 \, \Phi_{11} \; \overline{\rho} \; + 2 \, \Phi_{20} \; \sigma \; - \Phi_{02} \; \overline{\sigma} \; + 2 \, \Phi_{10} \; (\overline{\pi} \; + \tau \;) \; +$

$2 \, \Phi_{01} \; (\pi \; + \overline{\tau} \;) \; - 3 \, (\text{þ} \, \Psi_2 \;) \; + \text{þ}' \, \Phi_{00} \; + 2 \, (\text{ð} \, \Phi_{10} \;) \; - \text{ð}' \, \Phi_{01} \; + 3 \, (\text{ð}' \, \Psi_1 \;)),$

$\text{þ} \, \Phi_{12} \; == \; - \Phi_{22} \; \kappa \; - \Phi_{10} \; \overline{\lambda} \; - \Phi_{01} \; \mu \; + 2 \, \Psi_1 \; \mu \; - \Psi_0 \; \nu \; + \Phi_{02} \; \pi \; + 2 \, \Phi_{11} \; \overline{\pi} \; + \Phi_{12} \; \overline{\rho} \; +$

$\Phi_{21} \; \sigma \; - 2 \, \Psi_3 \; \sigma \; + 3 \, \Psi_2 \; \tau \; + \text{þ}' \, \Psi_1 \; - \text{ð} \, \Lambda \; + \text{ð} \, \Phi_{11} \; - \text{ð} \, \Psi_2 \; ,$

$\text{þ} \, \Phi_{21} \; == \; - 3 \, \Psi_4 \; \kappa \; - \Phi_{22} \; \overline{\kappa} \; + 2 \, \Phi_{01} \; \lambda \; - 6 \, \Psi_1 \; \lambda \; - 2 \, \Phi_{10} \; \mu \; + 2 \, \Phi_{10} \; \overline{\mu} \; - 2 \, \Phi_{00} \; \nu \; +$

$2 \, \Phi_{11} \; \pi \; + 9 \, \Psi_2 \; \pi \; + \Phi_{20} \; \overline{\pi} \; - 2 \, \Phi_{21} \; \rho \; + 6 \, \Psi_3 \; \rho \; + 2 \, \Phi_{21} \; \overline{\rho} \; - 2 \, \Phi_{12} \; \overline{\sigma} \; +$

$2 \, \Phi_{20} \; \tau \; + 4 \, \Phi_{11} \; \overline{\tau} \; - 3 \, (\text{þ} \, \Psi_3 \;) \; + 2 \, (\text{þ}' \, \Phi_{10} \;) \; + \text{ð} \, \Phi_{20} \; - 2 \, (\text{ð}' \, \Phi_{11} \;) \; + 3 \, (\text{ð}' \, \Psi_2 \;),$

$\text{þ} \, \Phi_{22} \; == \; 2 \, \Phi_{02} \; \lambda \; - \Phi_{20} \; \overline{\lambda} \; - 2 \, \Phi_{11} \; \mu \; + 9 \, \Psi_2 \; \mu \; + 4 \, \Phi_{11} \; \overline{\mu} \; - 2 \, (\Phi_{01} \; + 3 \, \Psi_1 \;) \; \nu \; -$

$2 \, \Phi_{10} \; \overline{\nu} \; - 2 \, \Phi_{22} \; \rho \; + \Phi_{22} \; \overline{\rho} \; - 3 \, \Psi_4 \; \sigma \; + 6 \, \Psi_3 \; \tau \; + 2 \, \Phi_{21} \; (\overline{\pi} \; + \tau \;) \; +$

$2 \, \Phi_{12} \; (\pi \; + \overline{\tau} \;) \; + 2 \, (\text{þ}' \, \Phi_{11} \;) \; + 3 \, (\text{þ}' \, \Psi_2 \;) \; + \text{ð} \, \Phi_{21} \; - 3 \, (\text{ð} \, \Psi_3 \;) \; - 2 \, (\text{ð}' \, \Phi_{12} \;), \; \text{þ} \, \Psi_4 \; ==$

$2 \, \Phi_{11} \; \lambda \; - 3 \, \Psi_2 \; \lambda \; + \Phi_{20} \; \overline{\mu} \; - 2 \, \Phi_{10} \; \nu \; + 4 \, \Psi_3 \; \pi \; + \Psi_4 \; \rho \; - \Phi_{22} \; \overline{\sigma} \; + 2 \, \Phi_{21} \; \overline{\tau} \; + \text{þ}' \, \Phi_{20} \; - \text{ð}' \, \Phi_{21} \; + \text{ð}' \, \Psi_3 \; ,$

$\text{þ}' \, \Lambda \; == \; - \Phi_{02} \; \lambda \; - 3 \, \Psi_2 \; \mu \; - 2 \, \Phi_{11} \; \overline{\mu} \; + (\Phi_{01} \; + 2 \, \Psi_1 \;) \; \nu \; + \Phi_{10} \; \overline{\nu} \; + \Phi_{22} \; \rho \; +$

$\Psi_4 \; \sigma \; - (\Phi_{21} \; + 2 \, \Psi_3 \;) \; \tau \; - \Phi_{12} \; \overline{\tau} \; - \text{þ}' \, \Phi_{11} \; - \text{þ}' \, \Psi_2 \; + \text{ð} \, \Psi_3 \; + \text{ð}' \, \Phi_{12} \; ,$

$\text{þ}' \, \Phi_{01} \; == \; - 2 \, \Psi_1 \; \mu \; - 2 \, \Phi_{01} \; \overline{\mu} \; + \Psi_0 \; \nu \; + \Phi_{00} \; \overline{\nu} \; + 2 \, \Phi_{12} \; \rho \; + 2 \, \Psi_3 \; \sigma \; - 2 \, \Phi_{11} \; \tau \; -$

$3 \, \Psi_2 \; \tau \; - \Phi_{02} \; \overline{\tau} \; - \text{þ}' \, \Psi_1 \; - 2 \, (\text{ð} \, \Lambda \;) \; + \text{ð} \, \Psi_2 \; + \text{ð}' \, \Phi_{02} \; , \; \text{þ}' \, \Phi_{21} \; ==$

$- 2 \, \Phi_{12} \; \lambda \; - 4 \, \Psi_3 \; \mu \; - 2 \, \Phi_{21} \; \overline{\mu} \; + 2 \, \Phi_{11} \; \nu \; + 3 \, \Psi_2 \; \nu \; + \Phi_{20} \; \overline{\nu} \; - \Psi_4 \; \tau \; - \Phi_{22} \; \overline{\tau} \; - \text{þ}' \, \Psi_3 \; + \text{ð} \, \Psi_4 \; + \text{ð}' \, \Phi_{22} \; \}$

Within the GHP formalism any expression must be weighted. The command GHPWeightOf will compute the weight of any expression.

In[382]:= **? GHPWeightOf**

GHPWeightOf[expr] returns the GHP boost and spin weight of any expression.
   If the expression is not weighted, then an error message is returned

For example: take one of the GHP Bianchi identities:

In[383]:= **GHPBianchi[[2]]**

Out[383]= $\text{þ} \, \Phi_{01} \; == \; - 2 \, \Phi_{11} \; \kappa \; - 3 \, \Psi_2 \; \kappa \; - \Phi_{02} \; \overline{\kappa} \; + \Psi_0 \; \pi \; + \Phi_{00} \; \overline{\pi} \; + 4 \, \Psi_1 \; \rho \; + 2 \, \Phi_{01} \; \overline{\rho} \; + 2 \, \Phi_{10} \; \sigma \; - \text{þ} \, \Psi_1 \; + \text{ð} \, \Phi_{00} \; + \text{ð}' \, \Psi_0$

We compute the GHP weight of the right hand side of this relation.

In[384]:= **GHPWeightOf@%[[2]]**

Out[384]= $\{3, 1\}$

Finally we explain how to compute the action of the GHP commutators on any quantity. For example suppose that we wish to compute the commutation of þ and ð' on the scalar Ψ_4 . First of all, we look up in the list of GHP commutators the commutation relation we wish to use. In our case this is

In[385]:= **GHPCommutators[[4]]**

Out[385]= $\text{þ} \, \text{ð'} \, \eta^{\{w, \, z\}} - \text{ð'} \, \text{þ} \, \eta^{\{w, \, z\}} == \eta^{\{w, \, z\}} \, \left(w \, (\Phi_{10} \, + \kappa \, \lambda \, - \pi \, \rho \,) - z \, \left(\overline{\Psi}_1 \, - \overline{\kappa} \, \overline{\mu} \, + \overline{\pi} \, \overline{\sigma} \, \right) \right) +$
$\pi \, \left(\text{þ} \, \eta^{\{w, \, z\}} \right) - \overline{\kappa} \, \left(\text{þ'} \, \eta^{\{w, \, z\}} \right) + \overline{\sigma} \, \left(\text{ð} \, \eta^{\{w, \, z\}} \right) + \rho \, \left(\text{ð'} \, \eta^{\{w, \, z\}} \right)$

Next we replace $\eta^{\{w, \, z\}}$ by Ψ_4 and the weights {w, z} by their values. This is accomplished as follows

In[386]:= **% /. {η@LI@{w, z} → PsiCDe4Dyad[],**
w → First@GHPWeightOf@PsiCDe4Dyad[], z → Last@GHPWeightOf@PsiCDe4Dyad[]}

Out[386]= $\text{þ} \, \text{ð'} \, \Psi_4 \, - \text{ð'} \, \text{þ} \, \Psi_4 \, == -4 \, \Psi_4 \, (\Phi_{10} \, + \kappa \, \lambda \, - \pi \, \rho \,) + \pi \, (\text{þ} \, \Psi_4 \,) - \overline{\kappa} \, (\text{þ'} \, \Psi_4 \,) + \overline{\sigma} \, (\text{ð} \, \Psi_4 \,) + \rho \, (\text{ð'} \, \Psi_4 \,)$