xAct‘Invar‘

Intro

`Invar‘` is currently a package for efficient simplification of the algebraic and differential polynomial scalars (usually known as *invariants*) of the Riemann tensor of a metric−compatible connection. The medium term goal is generalizing `Invar‘` to simplify generic polynomial expressions (with free indices) of the Riemann tensor.

`Invar‘` requires the tensor computer algebra system `xTensor‘` when running on *Mathematica*. There is a twin version for Maple, based on the tensor system `Canon`. It can be downloaded, under the General Public Licence, from

http://metric.iem.csic.es/Martin−Garcia/xAct/

http://www.lncc.br/~portugal/ (Maple version)

For further information, see the articles

The Invar tensor package, J.M. Martín−García, R. Portugal and L.R.U. Manssur, Comp. Phys. Comm. **177**, 640 **(**2007)

The Invar tensor package: Differential invariants of Riemann, J.M. Martín−García, D. Yllanes and R. Portugal, Comp. Phys. Comm. (2008)

Subsection 6.3 contains a number of tests designed to check most capabilities of `Invar‘`. They all should give zero.

Load the package and configure

`Invar‘` requires the free package `xTensor‘` to perform the underlying tensor and permutation manipulations. It can be downloaded from

http://metric.iem.csic.es/Martin−Garcia/xAct/

For a single−user installation under Linux the file xAct_<version>.tar.gz must be unpacked in the directory $Home/.−*Mathematica*/Applications giving a directory xAct. Then unpack the Invar.tar.gz file inside the xAct directory. We assume that configuration in the following. See the Readme file for more information, including installation instructions for other operating systems.

We first load the *Mathematica* kernel, and the `xTensor‘` package from the default directory

```
In[1]:=  mathRAM = MemoryInUse[]

Out[1]=  3059792
```

```
In[2]:= << xAct`xTensor`

        ------------------------------------------------------------------------------
          --
        Package xAct`xCore`  version 0.5.0, {2008, 5, 16}

        CopyRight (C) 2007-2008, Jose M.
          Martin-Garcia, under the General Public License.

        ------------------------------------------------------------------------------
          --
        Package ExpressionManipulation`

        CopyRight (C) 1999-2008, David J. M. Park and Ted Ersek

        ------------------------------------------------------------------------------
          --
        Package xAct`xPerm`  version 1.0.1, {2008, 5, 16}

        CopyRight (C) 2003-2008, Jose M.
          Martin-Garcia, under the General Public License.

        Connecting to external linux executable...

        Connection established.

        ------------------------------------------------------------------------------
          --
        Package xAct`xTensor`  version 0.9.5, {2008, 5, 16}

        CopyRight (C) 2002-2008, Jose M.
          Martin-Garcia, under the General Public License.

        ------------------------------------------------------------------------------
          --
        These packages come with ABSOLUTELY NO WARRANTY; for details type
          Disclaimer[]. This is free software, and you are welcome to redistribute
          it under certain conditions. See the General Public License for details.

        ------------------------------------------------------------------------------
          --
```

Note: When loading `xTensor`` inside a Windows environment, a black DOS−like window may appear. This is caused by the C link used to speed computations and should not be closed.

```
In[3]:= xtensorRAM = MemoryInUse[] - mathRAM

Out[3]= 12729104
```

Then we read the `Invar‘` package.

```
In[4]:= <<xAct‘Invar‘
```

```
            --------------------------------------------------------------------------------
              --
            Package xAct‘Invar‘  version 2.0.2, {2008, 3, 5}

            CopyRight (C) 2006-2008, J. M. Martin-Garcia, D.
              Yllanes and R. Portugal, under the General Public License.

            --------------------------------------------------------------------------------
              --
            These packages come with ABSOLUTELY NO WARRANTY; for details type
              Disclaimer[]. This is free software, and you are welcome to redistribute
              it under certain conditions. See the General Public License for details.

            --------------------------------------------------------------------------------
              --
            ** DefConstantSymbol: Defining constant symbol sigma.

            ** DefConstantSymbol: Defining constant symbol dim.
```

We see that, in *Mathematica* 5, the *MathKernel* takes only 2 Mbytes, `xTensor‘` takes 9 Mbytes, and `Invar‘` takes 41 Mbytes:

```
In[5]:= invarRAM = MemoryInUse[] - mathRAM - xtensorRAM

Out[5]= 299616

In[6]:= Remove[mathRAM, xtensorRAM, invarRAM]
```

Note the structure of the ContextPath. There are six contexts: `xAct‘Invar‘`, `xAct‘xTensor‘`, `xAct‘xPerm‘` and `xAct‘ExpressionManipulation‘` contain the respective reserved words. `System‘` contains *Mathematica*'s reserved words. The current context `Global‘` will contain your definitions and right now it is empty.

```
In[7]:= $ContextPath

Out[7]= {xAct‘Invar‘, xAct‘xTensor‘, xAct‘xPerm‘,
         xAct‘xCore‘, xAct‘ExpressionManipulation‘, Global‘, System‘}

In[8]:= Context[]

Out[8]= Global‘

In[9]:= ?Global‘*

        Information::nomatch : No symbol matching Global‘* found. More…
```

■ 1. Example session

This is a simple example of the use of `Invar‘`. The package is very easy to use: all its capabilities can be controlled through 10 basic commands. In addition, you will have to setup a basic `xTensor‘` session, which we proceed to do now. We need to start by defining the manifold and metric of the Riemann tensor. For information on `xTensor‘` see the file xTensorDoc.nb.

Define a 4d manifold M:

```
In[10]:= DefManifold[M, {1, 2, 3, 4}, {a, b, c, d, e, f, g, h, i, j, k, l, m, n}]

         ** DefManifold: Defining manifold M.

         ** DefVBundle: Defining vbundle TangentM.
```

Define a metric with negative determinant, with associated Levi−Civita connection CD:

```
In[11]:= DefMetric[-1, metric[-a, -b], CD, {";", "∇"}, CurvatureRelations → False]

         ** DefTensor: Defining symmetric metric tensor metric[-a, -b].

         ** DefTensor: Defining antisymmetric tensor epsilonmetric[a, b, c, d].

         ** DefCovD: Defining covariant derivative CD[-a].

         ** DefTensor: Defining vanishing torsion tensor TorsionCD[a, -b, -c].

         ** DefTensor: Defining symmetric Christoffel tensor ChristoffelCD[a, -b, -c].

         ** DefTensor: Defining Riemann tensor RiemannCD[-a, -b, -c, -d].

         ** DefTensor: Defining symmetric Ricci tensor RicciCD[-a, -b].

         ** DefTensor: Defining Ricci scalar RicciScalarCD[].

         ** DefTensor: Defining symmetric Einstein tensor EinsteinCD[-a, -b].

         ** DefTensor: Defining Weyl tensor WeylCD[-a, -b, -c, -d].

            Rules {1, 2, 3, 4, 5, 6, 7, 8} have been declared as DownValues for WeylCD.

         ** DefTensor: Defining symmetric TFRicci tensor TFRicciCD[-a, -b].

            Rules {1, 2} have been declared as DownValues for TFRicciCD.

         ** DefCovD:  Computing RiemannToWeylRules for dim 4

         ** DefCovD:  Computing RicciToTFRicci for dim 4

         ** DefCovD:  Computing RicciToEinsteinRules for dim 4
```

We included the option CurvatureRelations −> False so contractions of Riemanns will not be converted into Riccis. A number of tensors have been automatically defined. In particular there are the three curvature tensor fields

```
In[12]:= RiemannCD[-a, -b, -c, -d]
```

Out[12]= $R[\nabla]_{abcd}$

```
In[13]:= RicciCD[-a, -b]
```

Out[13]= $R[\nabla]_{ab}$

```
In[14]:= RicciScalarCD[]
```

Out[14]= $R[\nabla]$

and their traceless counterparts:

```
In[15]:= WeylCD[-a, -b, -c, -d]
```

Out[15]= $W[\nabla]_{abcd}$

```
In[16]:= TFRicciCD[-a, -b]
```

Out[16]= $S[\nabla]_{ab}$

with their expected trace properties:

```
In[17]:= WeylCD[a, -b, -c, -a]

Out[17]= 0

In[18]:= TFRicciCD[-a, a]

Out[18]= 0
```

The Riemann and Ricci tensors are not automatically replaced when they have contracted indices. Use **ContractCurvature** to do that:

```
In[19]:= RiemannCD[a, -b, -c, -a]
```

Out[19]= $R[\nabla]^{a}{}_{bca}$

```
In[20]:= ContractCurvature[%]
```

Out[20]= $-R[\nabla]_{bc}$

```
In[21]:= RicciCD[-b, b]
```

Out[21]= $R[\nabla]_{b}{}^{b}$

```
In[22]:= ContractCurvature[%]
```

Out[22]= $R[\nabla]$

We can change the output form of the tensors (note the use of *Mathematica* downvalues ^=)

```
In[23]:= PrintAs[metric] ^= "g";
         PrintAs[epsilonmetric] ^= "ϵ";
         PrintAs[RiemannCD] ^= "R";
         PrintAs[RicciCD] ^= "R";
         PrintAs[RicciScalarCD] ^= "R";
         PrintAs[WeylCD] ^= "W";
         PrintAs[TFRicciCD] ^= "S";
```

So much for `xTensor‘`. You will not need many more `xTensor‘` functions to work with `Invar‘`, but you will be able to do it more efficiently if you read the specific documentation for that package. Now we can start using the commands added by the package `Invar‘`:

A typical algebraic Riemann scalar could be

```
In[30]:= RiemannCD[a, b, -c, e] RiemannCD[-b, -e, -f, d] RiemannCD[c, f, -a, -d]
```

Out[30]= $R^{ab}{}_{c}{}^{e}\, R_{bef}{}^{d}\, R^{cf}{}_{ad}$

which simplifies to

```
In[31]:= RiemannSimplify[%] // Timing

           Reading InvRules for step 1 and case {0, 0, 0}

           Reading InvRules for step 2 and case {0, 0, 0}

           Reading InvRules for step 3 and case {0, 0, 0}

           Reading InvRules for step 4 and case {0, 0, 0}

           Reading InvRules for step 5, case {0, 0, 0} and dimension 4

           Reading InvRules for step 6, case {0, 0, 0} and dimension 4
```

Out[31]= $\left\{0.084005\,\text{Second},\ -\frac{1}{4}\,\text{Scalar}[R_{ab}{}^{ef}\, R^{abcd}\, R_{cdef}]\right\}$

Notice how the different rules are read on the fly. We do not read them automatically because they would need several GB of memory. They are read only once (note the shorter timing now):

```
In[32]:= RiemannSimplify[%%] // Timing
```

Out[32]= $\left\{0.024001\,\text{Second},\ -\frac{1}{4}\,\text{Scalar}[R_{ab}{}^{ef}\, R^{abcd}\, R_{cdef}]\right\}$

The head `Scalar` is introduced to avoid problems of index collisions.It can be removed using `NoScalar`:

```
In[33]:= Last[%] // NoScalar
```

Out[33]= $-\frac{1}{4}\, R_{ab}{}^{ef}\, R^{abcd}\, R_{cdef}$

We can generate random Riemann invariants using the command `RandomRiemannMonomial`, having as argument the case (see Section 2), in this example {0,4}, meaning one Riemann with no derivatives and another with a fourth derivative:

```
In[34]:= expr = RandomRiemannMonomial[{0, 4}] // NoScalar
```

Out[34]= $R^{ea}{}_{a}{}^{d}\, R^{c}{}_{c}{}^{f}{}_{b}{}^{;b}{}_{;f;e;d}$

Any monomial can be canonicalized into one of a predefined list of canonical invariants. For the case {0,4} the list contains 126 canonical invariants, apart from the trivial canonical 0.

```
In[35]:= MaxIndex[{0, 4}]

Out[35]= 126
```

In particular, the previous object canonicalizes to

```
In[36]:=  RiemannToInv[expr]

Out[36]=  0
```

Many of the invariants actually do canonicalize to 0 so we iterate those two commands until we get something different from zero:

```
In[37]:=  inv = 0;
          While[inv === 0,
            expr = RandomRiemannMonomial[{0, 4}] // NoScalar;
            inv = RiemannToInv[expr];
            Print[ScreenDollarIndices[expr], " -> ", inv]];

            Reading InvRules for step 1 and case {0, 4}
```

$R^{ce}{}_{bd}\, R^{ba}{}_{f}{}^{d}{}_{;a;e}{}^{;f}{}_{;c}$ -> $I_{04,113}$

```
In[39]:=  RiemannToInv[expr]
```

Out[39]= $I_{04,113}$

The canonical invariants can be expressed as a polynomial in a basis of indepent invariants, taking into account all the symmetries of the problem. This implies reading rules for additional 'steps'

```
In[40]:= InvSimplify[%]

             Reading InvRules for step 2 and case {0, 4}

             Reading InvRules for step 3 and case {0, 4}

             Reading InvRules for step 4 and case {0, 4}

             Reading InvRules for step 5, case {0, 4} and dimension 4

             Reading InvRules for step 6, case {0, 4} and dimension 4

             Reading InvRules for step 5, case {0, 0, 2} and dimension 4

             Reading InvRules for step 5, case {0, 1, 1} and dimension 4

             Reading InvRules for step 5, case {0, 0, 0, 0} and dimension 4

             Reading InvRules for step 6, case {0} and dimension 4

             Reading InvRules for step 6, case {2} and dimension 4

             Reading InvRules for step 6, case {0, 0} and dimension 4

             Reading InvRules for step 6, case {0, 2} and dimension 4

             Reading InvRules for step 6, case {0, 0, 2} and dimension 4

             Reading InvRules for step 6, case {0, 1, 1} and dimension 4

             Reading InvRules for step 6, case {1, 1} and dimension 4

             Reading InvRules for step 6, case {0, 0, 0, 0} and dimension 4
```

Out[40]=

$$-\frac{5}{16} I_{0,1}^2 I_{2,1} + \frac{1}{8} I_{0,1}^2 I_{00,1} + \frac{3 I_{2,1} I_{00,1}}{4} - \frac{I_{00,1}^2}{2} - \frac{I_{2,1} I_{00,2}}{16} + \frac{I_{00,1} I_{00,2}}{8} + \frac{3 I_{0,1} I_{02,2}}{2} - \frac{I_{0,1} I_{02,6}}{2} + \frac{I_{0,1} I_{11,1}}{8} - \frac{I_{0,1} I_{11,5}}{2} + 2 I_{0,1} I_{000,1} + \frac{3 I_{0,1} I_{000,2}}{2} + 2 I_{002,2} - I_{002,5} + I_{002,6} - I_{002,8} - I_{002,14} + I_{002,52} - \frac{I_{011,2}}{2} + \frac{I_{011,3}}{2} - I_{011,9} - I_{011,10} + I_{011,18} + I_{011,24} + I_{011,25} + \frac{I_{011,28}}{8} - I_{011,58} + 3 I_{0000,1} - \frac{I_{0000,5}}{2} - I_{0000,7}$$

Finally this expression can be translated back into Riemann invariants.

```
In[41]:= InvToRiemann[%]

           Reading InvRules for step 1 and case {0}
           Reading InvRules for step 1 and case {2}
           Reading InvRules for step 1 and case {0, 0}
           Reading InvRules for step 1 and case {0, 2}
           Reading InvRules for step 1 and case {1, 1}
           Reading InvRules for step 1 and case {0, 0, 2}
           Reading InvRules for step 1 and case {0, 1, 1}
           Reading InvRules for step 1 and case {0, 0, 0, 0}
```

Out[41]=

$$\begin{aligned}
&\frac{1}{8}\,\text{Scalar}[R^{ab}{}_{ab}]^2\,\text{Scalar}[R^{ab}{}_a{}^c\,R_b{}^d{}_{cd}] + \frac{1}{8}\,\text{Scalar}[R_{abcd}\,R^{abcd}]\,\text{Scalar}[R^{ab}{}_a{}^c\,R_b{}^d{}_{cd}] - \\
&\frac{1}{2}\,\text{Scalar}[R^{ab}{}_a{}^c\,R_b{}^d{}_{cd}]^2 + 2\,\text{Scalar}[R^{ab}{}_{ab}]\,\text{Scalar}[R^{ab}{}_a{}^c\,R_b{}^d{}_d{}^e\,R_c{}^f{}_{ef}] + \\
&\frac{3}{2}\,\text{Scalar}[R^{ab}{}_{ab}]\,\text{Scalar}[R^{ab}{}_a{}^c\,R_b{}^d{}_c{}^e\,R_d{}^f{}_{ef}] - \frac{1}{2}\,\text{Scalar}[R^{ab}{}_a{}^c\,R_b{}^{def}\,R_c{}^g{}_{ef}\,R_d{}^h{}_{gh}] + \\
&3\,\text{Scalar}[R^{ab}{}_a{}^c\,R_b{}^d{}_d{}^e\,R_c{}^f{}_f{}^g\,R_e{}^h{}_{gh}] - \text{Scalar}[R^{ab}{}_a{}^c\,R_b{}^d{}_c{}^e\,R_d{}^f{}_e{}^g\,R_f{}^h{}_{gh}] - \\
&\frac{5}{16}\,\text{Scalar}[R^{ab}{}_{ab}]^2\,\text{Scalar}[R^{ab}{}_{ab}{}^{;c}{}_{;c}] - \frac{1}{16}\,\text{Scalar}[R_{abcd}\,R^{abcd}]\,\text{Scalar}[R^{ab}{}_{ab}{}^{;c}{}_{;c}] + \\
&\frac{3}{4}\,\text{Scalar}[R^{ab}{}_a{}^c\,R_b{}^d{}_{cd}]\,\text{Scalar}[R^{ab}{}_{ab}{}^{;c}{}_{;c}] + \frac{1}{8}\,\text{Scalar}[R^{ab}{}_{ab}]\,\text{Scalar}[R^{de}{}_{de;c}\,R^{ab}{}_{ab}{}^{;c}] + \\
&\text{Scalar}[R^{abcd}\,R_a{}^e{}_e{}^f{}_{;b}\,R_c{}^g{}_{fg;d}] - \frac{1}{2}\,\text{Scalar}[R^{ab}{}_{ab}]\,\text{Scalar}[R^{abcd}\,R_a{}^e{}_{ce;b;d}] - \\
&\frac{1}{2}\,\text{Scalar}[R^{ab}{}_{ab}]\,\text{Scalar}[R_b{}^e{}_{de;c}\,R^{ab}{}_a{}^{c;d}] + \frac{1}{2}\,\text{Scalar}[R^{ab}{}_a{}^c\,R_b{}^d{}_d{}^e{}_{;c}\,R^{fg}{}_{fg;e}] + \\
&\frac{3}{2}\,\text{Scalar}[R^{ab}{}_{ab}]\,\text{Scalar}[R^{ab}{}_a{}^c\,R_b{}^d{}_{cd}{}^{;e}{}_{;e}] + \frac{1}{8}\,\text{Scalar}[R^{abcd}\,R^{fg}{}_{fg;e}\,R_{abcd}{}^{;e}] - \\
&\frac{1}{2}\,\text{Scalar}[R^{ab}{}_a{}^c\,R^{fg}{}_{fg;e}\,R_b{}^d{}_{cd}{}^{;e}] + \text{Scalar}[R^{ab}{}_a{}^c\,R^{de}{}_d{}^f\,R_b{}^g{}_{eg;c;f}] + \text{Scalar}[R_{ab}{}^{ef}\,R^{abcd}\,R_c{}^g{}_{eg;d;f}] - \\
&\text{Scalar}[R^{ab}{}_a{}^c\,R_b{}^{def}\,R_c{}^g{}_{eg;d;f}] - \text{Scalar}[R^{ab}{}_a{}^c\,R^{de}{}_d{}^f\,R_b{}^g{}_{cg;e;f}] - \text{Scalar}[R^{abcd}\,R_d{}^g{}_{fg;e}\,R_{abc}{}^{e;f}] + \\
&\text{Scalar}[R^{abcd}\,R_b{}^g{}_{fg;d}\,R_a{}^e{}_{ce}{}^{;f}] + \text{Scalar}[R^{ab}{}_a{}^c\,R_d{}^g{}_{fg;e}\,R_b{}^d{}_c{}^{e;f}] - \text{Scalar}[R^{ab}{}_a{}^c\,R_e{}^g{}_{fg;c}\,R_b{}^d{}_d{}^{e;f}] - \\
&\text{Scalar}[R^{ab}{}_a{}^c\,R_c{}^g{}_{fg;e}\,R_b{}^d{}_d{}^{e;f}] + 2\,\text{Scalar}[R^{ab}{}_a{}^c\,R_b{}^d{}_d{}^e\,R_c{}^f{}_{ef}{}^{;g}{}_{;g}] - \text{Scalar}[R^{ab}{}_a{}^c\,R_b{}^d{}_c{}^e\,R_d{}^f{}_{ef}{}^{;g}{}_{;g}]
\end{aligned}$$

The whole process can be done in a single step using the command RiemannSimplify:

```
In[42]:= RiemannSimplify[expr]
```

Out[42]=

$$\begin{aligned}
&\frac{1}{8}\,\text{Scalar}[R^{ab}{}_{ab}]^2\,\text{Scalar}[R^{ab}{}_{a}{}^{c}\,R_{b}{}^{d}{}_{cd}] + \frac{1}{8}\,\text{Scalar}[R_{abcd}\,R^{abcd}]\,\text{Scalar}[R^{ab}{}_{a}{}^{c}\,R_{b}{}^{d}{}_{cd}] - \\
&\frac{1}{2}\,\text{Scalar}[R^{ab}{}_{a}{}^{c}\,R_{b}{}^{d}{}_{cd}]^2 + 2\,\text{Scalar}[R^{ab}{}_{ab}]\,\text{Scalar}[R^{ab}{}_{a}{}^{c}\,R_{b}{}^{d}{}_{d}{}^{e}\,R_{c}{}^{f}{}_{ef}] + \\
&\frac{3}{2}\,\text{Scalar}[R^{ab}{}_{ab}]\,\text{Scalar}[R^{ab}{}_{a}{}^{c}\,R_{b}{}^{d}{}_{c}{}^{e}\,R_{d}{}^{f}{}_{ef}] - \frac{1}{2}\,\text{Scalar}[R^{ab}{}_{a}{}^{c}\,R_{b}{}^{def}\,R_{c}{}^{g}{}_{ef}\,R_{d}{}^{h}{}_{gh}] + \\
&3\,\text{Scalar}[R^{ab}{}_{a}{}^{c}\,R_{b}{}^{d}{}_{d}{}^{e}\,R_{c}{}^{f}{}_{f}{}^{g}\,R_{e}{}^{h}{}_{gh}] - \text{Scalar}[R^{ab}{}_{a}{}^{c}\,R_{b}{}^{d}{}_{c}{}^{e}\,R_{d}{}^{f}{}_{e}{}^{g}\,R_{f}{}^{h}{}_{gh}] - \\
&\frac{5}{16}\,\text{Scalar}[R^{ab}{}_{ab}]^2\,\text{Scalar}[R^{ab}{}_{ab}{}^{;c}{}_{;c}] - \frac{1}{16}\,\text{Scalar}[R_{abcd}\,R^{abcd}]\,\text{Scalar}[R^{ab}{}_{ab}{}^{;c}{}_{;c}] + \\
&\frac{3}{4}\,\text{Scalar}[R^{ab}{}_{a}{}^{c}\,R_{b}{}^{d}{}_{cd}]\,\text{Scalar}[R^{ab}{}_{ab}{}^{;c}{}_{;c}] + \frac{1}{8}\,\text{Scalar}[R^{ab}{}_{ab}]\,\text{Scalar}[R^{de}{}_{de;c}\,R^{ab}{}_{ab}{}^{;c}] + \\
&\text{Scalar}[R^{abcd}\,R_{a}{}^{e}{}_{e}{}^{f}{}_{;b}\,R_{c}{}^{g}{}_{fg;d}] - \frac{1}{2}\,\text{Scalar}[R^{ab}{}_{ab}]\,\text{Scalar}[R^{abcd}\,R_{a}{}^{e}{}_{ce;b;d}] - \\
&\frac{1}{2}\,\text{Scalar}[R^{ab}{}_{ab}]\,\text{Scalar}[R_{b}{}^{e}{}_{de;c}\,R^{ab}{}_{a}{}^{c;d}] + \frac{1}{2}\,\text{Scalar}[R^{ab}{}_{a}{}^{c}\,R_{b}{}^{d}{}_{d}{}^{e}{}_{;c}\,R^{fg}{}_{fg;e}] + \\
&\frac{3}{2}\,\text{Scalar}[R^{ab}{}_{ab}]\,\text{Scalar}[R^{ab}{}_{a}{}^{c}\,R_{b}{}^{d}{}_{cd}{}^{;e}{}_{;e}] + \frac{1}{8}\,\text{Scalar}[R^{abcd}\,R^{fg}{}_{fg;e}\,R_{abcd}{}^{;e}] - \\
&\frac{1}{2}\,\text{Scalar}[R^{ab}{}_{a}{}^{c}\,R^{fg}{}_{fg;e}\,R_{b}{}^{d}{}_{cd}{}^{;e}] + \text{Scalar}[R^{ab}{}_{a}{}^{c}\,R^{de}{}_{d}{}^{f}\,R_{b}{}^{g}{}_{eg;c;f}] + \text{Scalar}[R_{ab}{}^{ef}\,R^{abcd}\,R_{c}{}^{g}{}_{eg;d;f}] - \\
&\text{Scalar}[R^{ab}{}_{a}{}^{c}\,R_{b}{}^{def}\,R_{c}{}^{g}{}_{eg;d;f}] - \text{Scalar}[R^{ab}{}_{a}{}^{c}\,R^{de}{}_{d}{}^{f}\,R_{b}{}^{g}{}_{cg;e;f}] - \text{Scalar}[R^{abcd}\,R_{d}{}^{g}{}_{fg;e}\,R_{abc}{}^{e;f}] + \\
&\text{Scalar}[R^{abcd}\,R_{b}{}^{g}{}_{fg;d}\,R_{a}{}^{e}{}_{ce}{}^{;f}] + \text{Scalar}[R^{ab}{}_{a}{}^{c}\,R_{d}{}^{g}{}_{fg;e}\,R_{b}{}^{d}{}_{c}{}^{e;f}] - \text{Scalar}[R^{ab}{}_{a}{}^{c}\,R_{e}{}^{g}{}_{fg;c}\,R_{b}{}^{d}{}_{d}{}^{e;f}] - \\
&\text{Scalar}[R^{ab}{}_{a}{}^{c}\,R_{c}{}^{g}{}_{fg;e}\,R_{b}{}^{d}{}_{d}{}^{e;f}] + 2\,\text{Scalar}[R^{ab}{}_{a}{}^{c}\,R_{b}{}^{d}{}_{d}{}^{e}\,R_{c}{}^{f}{}_{ef}{}^{;g}{}_{;g}] - \text{Scalar}[R^{ab}{}_{a}{}^{c}\,R_{b}{}^{d}{}_{c}{}^{e}\,R_{d}{}^{f}{}_{ef}{}^{;g}{}_{;g}]
\end{aligned}$$

This is the example expression shown in the first paper:

```
In[43]:= expr = epsilonmetric[a, b, c, d] RiemannCD[-a, -b, e, f]
            RiemannCD[-c, -e, -f, g] RiemannCD[-d, h, i, j] RicciCD[-g, -i] RicciCD[-h, -j] +
           1 / 8 epsilonmetric[a, b, c, d] RiemannCD[-a, -b, e, f] RiemannCD[-c, -d, -e, -f]
            RiemannCD[g, h, i, j] RicciCD[-g, -i] RicciCD[-h, -j]
```

Out[43]= $\epsilon^{abcd}\,R_{gi}\,R_{hj}\,R_{ab}{}^{ef}\,R_{cef}{}^{g}\,R_{d}{}^{hij} + \frac{1}{8}\,\epsilon^{abcd}\,R_{gi}\,R_{hj}\,R_{ab}{}^{ef}\,R_{cdef}\,R^{ghij}$

```
In[44]:= RiemannSimplify[expr] // Timing

            Reading DualInvRules for step 1 and case {0, 0}

            Reading DualInvRules for step 1 and case {0, 0, 0, 0, 0}

            Reading DualInvRules for step 2 and case {0, 0, 0, 0, 0}

            Reading DualInvRules for step 3 and case {0, 0, 0, 0, 0}

            Reading DualInvRules for step 4 and case {0, 0, 0, 0, 0}

            Reading DualInvRules for step 5 and case {0, 0, 0, 0, 0}

            Reading DualInvRules for step 2 and case {0, 0}

            Reading DualInvRules for step 3 and case {0, 0}

            Reading DualInvRules for step 4 and case {0, 0}

            Reading DualInvRules for step 5 and case {0, 0}

Out[44]= {0.524032 Second, 0}
```

And this is the example expression shown in the second paper:

```
In[45]:= expr = CD[-a]@CD[e]@RiemannCD[a, b, c, d] CD[i]@CD[h]@RiemannCD[-b, -e, f, g]
            CD[-h]@CD[-d]@RiemannCD[-c, -f, -g, -i] - 1/8 CD[-e]@CD[e]@RiemannCD[a, b, c, d]
            CD[i]@CD[h]@RiemannCD[-a, -b, f, g] CD[-h]@CD[-i]@RiemannCD[-c, -d, -f, -g]
```

Out[45]= $-\frac{1}{8} R^{abcd;e}{}_{;e}\, R_{cdfg;i;h}\, R_{ab}{}^{fg;h;i} + R^{abcd;e}{}_{;a}\, R_{cfgi;d;h}\, R_{be}{}^{fg;h;i}$

```
In[46]:= RiemannSimplify[expr] // Timing

           Reading InvRules for step 1 and case {2, 2, 2}

           Reading InvRules for step 2 and case {2, 2, 2}

           Reading InvRules for step 3 and case {2, 2, 2}

           Reading NEInvRules for step 4 and case {2, 2, 2}

           Reading InvRules for step 5, case {2, 2, 2} and dimension 4

           Reading InvRules for step 6, case {2, 2, 2} and dimension 4

           Reading InvRules for step 5, case {0, 0, 2, 2} and dimension 4

           Reading InvRules for step 6, case {0, 0, 2, 2} and dimension 4

Out[46]= {1.72411 Second, 0}
```

■ 2. General concepts and definitions

The Riemann tensor obeys two types of symmetries. The first type are *permutation* (or *monoterm*) symmetries, for example,

$$R_{\mathrm{bacd}} = -R_{\mathrm{abcd}} \qquad \text{and} \qquad R_{\mathrm{cdab}} = R_{\mathrm{abcd}} \ .$$

The second type is the *cyclic* (or *multiterm*) symmetry:

$$R_{\mathrm{abcd}} + R_{\mathrm{acdb}} + R_{\mathrm{adbc}} = 0\ .$$

In order to simplify any polynomial of the Riemann tensor, it is necessary to use both kinds. It is possible to use efficient algorithms of computational group theory to manipulate the former, but there are no known efficient algorithms to manipulate the latter.

If we include covariant derivatives, we have to consider two new classes of relations. We have the Bianchi identity:

$$R_{ab[cd;e]} = 0$$

and the effect of non commutatity of covariant derivatives:

$$\nabla_d \nabla_c T^{a_1 \dots a_n}{}_{b_1 \dots b_n} - \nabla_c \nabla_d T^{a_1 \dots a_n}{}_{b_1 \dots b_m} = \sum_{k=1}^n R_{cde}{}^{a_k}\, T^{a_1 \dots e \dots a_n}{}_{b_1 \dots b_m} - \sum_{k=1}^n R_{cdb_k}{}^{e}\, T^{a_1 \dots a_n}{}_{b_1 \dots e \dots b_m}$$

An additional problem is that there are more identities to consider, coming from the fact that some of the polynomial invariants are zero in low−enough dimension. It is possible to show that all these dimension dependent identities are of Lovelock type, that is, they all come from the idea that, on a manifold of dimension *dim*, antisymmetrizing in *dim* + 1 indices gives zero. Unfortunately, these last three types of identities cannot be manipulated efficiently either.

The main idea behind `Invar`` is using the fast canonicalizers of `xTensor`` and `Canon` to deal with the permutation symmetries, and then using prestored files of solutions for the other types of identities.

The package is structured as follows:

1. We shall work with two types of invariants:

− (*non−dual*) invariants: fully contracted products of n Riemann tensors (where n will be called the *degree* of the invariant), possibly including covariant derivatives .

− *dual* invariants: fully contracted products of n Riemann tensors with or without derivatives and one epsilon tensor (n is again called the *degree* of the dual invariant). There is no need to work with more than one epsilon, because products of two epsilons can be converted into delta tensors.

We shall separate the set of all monomial invariants of degree n in subsets $\mathcal{R}_{\{\lambda_1, \ldots, \lambda_n\}}$, where λ_i is the differentiation order of the i−th Riemann tensor, assuming the tensors have sorted such that $\lambda_i \leq \lambda_{i+}$. An n−tuple $\{\lambda_1, \ldots, \lambda_n\}$ will be referred to as a *case*, with the corresponding invariants having $N = 4n + \sum_{i=1}^{n} \lambda_i$ indices and order $\Lambda = 2n + \sum_{i=1}^{n} \lambda_i$. Λ is the number of derivatives of the metric, not to be confused with the total number of derivatives of the Riemann tensors (the sum of the λ_i's). Both N and Λ are always even numbers because we only consider scalar expressions. For example, an invariant of the case {0,1,3}, hence with $N = 16$ indices and order $\Lambda = 10$, is

$$R_{abcd} \nabla_e R^{ecfg} \nabla^a \nabla_f \nabla_h R^{bdh}{}_g .$$

2. We shall distinguish six steps or levels of simplification:

1.− Canonicalization with respect to permutation symmetries.

2.− Canonicalization with respect to the cyclic symmetry.

3.− Canonicalization with respect to the Bianchi symmetry.

4.− Canonicalization with respect to commutation of derivatives.

5.− Use of all possible dimensionally−dependent identities.

6.− Conversion of invariants into products of two dual invariants (metric−signature dependent).

3. Each Riemann monomial can be represented in one of three forms (examples will be given below):

− As an explicit tensor expression.

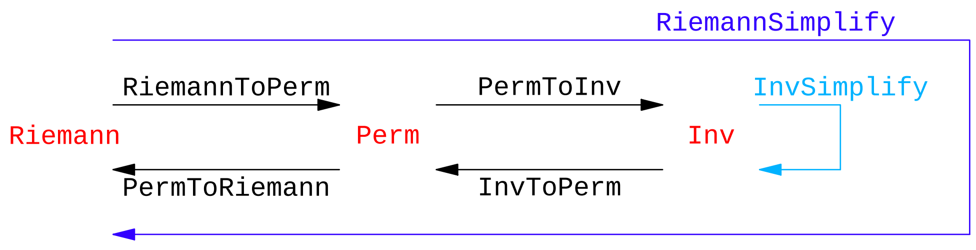
− As a permutation of the indices in canonical order.

− As an expression `RInv[metric][case, index]` or `DualRInv[metric][case, index]` which represents the equivalent canonical monomial after canonicalization with respect to the permutation symmetries. The integer index identifies each particular invariant in the list of invariants, sorted according to some predefined ordering.

4. Commands are given to change among those three representations, and to simplify the invariants up to some particular level. They are all listed below. Their mutual structure is schematically given by

```
In[47]:= << Graphics`Arrow`
```

```
In[48]:= SetOptions[Arrow, HeadLength → 0.02];
         os = 0.15;
         Show[
          Graphics[
           {Hue[0], Text["Riemann", {0, 0}], Text["Perm", {1, 0}], Text["Inv", {2, 0}]}],
          Graphics[{Text["RiemannToPerm", {0.5, 0.15}], Text["PermToInv", {1.5, 0.15}],
            Text["InvToPerm", {1.5, -0.16}], Text["PermToRiemann", {0.5, -0.16}]}],
          Graphics[{Arrow[{os, 0.1}, {1 - os, 0.1}], Arrow[{1 + os, 0.1}, {2 - os, 0.1}],
            Arrow[{1 - os, -0.1}, {os, -0.1}], Arrow[{2 - os, -0.1}, {1 + os, -0.1}]}],
          Graphics[{Hue[0.55], Line[{{2 + os, 0.1}, {2.4, 0.1}, {2.4, -0.1}}],
            Arrow[{2.4, -0.1}, {2 + os, -0.1}], Text["InvSimplify", {2.4, 0.15}]}],
          Graphics[{Hue[0.7], Line[{{os, 0.3}, {2.8, 0.3}, {2.8, -0.3}}],
            Arrow[{2.8, -0.3}, {os, -0.3}], Text["RiemannSimplify", {2.2, 0.35}]}],
          PlotRange → {{-0.3, 3}, {-0.5, 0.5}}, AspectRatio → Automatic, ImageSize → 400]
```



```
Out[50]= - Graphics -
```

`Invar'` currently handles invariants up to and including $\Lambda = 12$, plus the single case {0, 0, 0, 0, 0, 0, 0} of order 14. The full list of cases can be generated with the following command,

```
In[51]:= InvarCases[]

Out[51]= {{0}, {0, 0}, {2}, {0, 0, 0}, {0, 2}, {1, 1}, {4}, {0, 0, 0, 0}, {0, 0, 2}, {0, 1, 1},
          {0, 4}, {1, 3}, {2, 2}, {6}, {0, 0, 0, 0, 0}, {0, 0, 0, 2}, {0, 0, 1, 1}, {0, 0, 4},
          {0, 1, 3}, {0, 2, 2}, {1, 1, 2}, {0, 6}, {1, 5}, {2, 4}, {3, 3}, {8}, {0, 0, 0, 0, 0, 0},
          {0, 0, 0, 0, 2}, {0, 0, 0, 1, 1}, {0, 0, 0, 4}, {0, 0, 1, 3}, {0, 0, 2, 2}, {0, 1, 1, 2},
          {1, 1, 1, 1}, {0, 0, 6}, {0, 1, 5}, {0, 2, 4}, {1, 1, 4}, {0, 3, 3}, {1, 2, 3},
          {2, 2, 2}, {0, 8}, {1, 7}, {2, 6}, {3, 5}, {4, 4}, {10}, {0, 0, 0, 0, 0, 0, 0}}
```

This function admits two additional arguments. The first specifies the order

```
In[52]:= InvarCases[8]

Out[52]= {{0, 0, 0, 0}, {0, 0, 2}, {0, 1, 1}, {0, 4}, {1, 3}, {2, 2}, {6}}
```

and the second specifies the degree

```
In[53]:= InvarCases[8, 3]

Out[53]= {{0, 0, 2}, {0, 1, 1}}
```

3. Canonical invariants

`RInv`	Non−dual invariant
`DualRInv`	Dual invariant
`RInvs`	List of all non−dual invariants after a given step
`DualRInvs`	List of all dual invariants after a given step
`MaxIndex`	Maximum value of the index for the list of non−dual invariant after a given step
`MaxDualIndex`	Maximum value of the index for the list of dual invariants after a given step
`RInvRules`	List of all independent invariants for a given step and case
`RemoveRInvRules`	Remove rules for invariants of a given step and case

Canonical invariants

For a given case $\{\lambda_1, \ ..., \ \lambda_n\}$, there will be $N!$ different monomials, where $N = 4n + \sum_{i=1}^{n} \lambda_i$, as we saw in the previous section. It is obviously impossible to store them all.

Consider, for example, the numbers of possible permutations taking into acount only algebraic invariants, for wich the number of indices is N = 4n:

```
In[54]:= TableForm[Table[{n, (4 n) !, N[(4 n) !]}, {n, 7}],
           TableHeadings → {None, {"degree", "number of monomials"}}]

 Out[54]//TableForm=
         degree      number of monomials
         1           24                                  24.
         2           40320                               40320.
         3           479001600                           4.79002 × 10^8
         4           20922789888000                      2.09228 × 10^13
         5           2432902008176640000                 2.4329 × 10^18
         6           620448401733239439360000            6.20448 × 10^23
         7           304888344611713860501504000000      3.04888 × 10^29
```

We can, however, store the corresponding canonical invariants with respect to permutation symmetries. The numbers of canonical invariants for each case are stored in the functions `MaxIndex` and `MaxDualIndex`,

```
In[55]:= MaxIndex[{0, 2, 2}]

Out[55]= 1622
```

For example, the values up to order 10 are

```
In[56]:= MatrixForm[
          {InvarCases[][[Range[26]]], MaxIndex /@ InvarCases[][[Range[26]]]} // Transpose]

  Out[56]//MatrixForm=
         ⎛{0}              1    ⎞
         ⎜{0, 0}           3    ⎟
         ⎜{2}              2    ⎟
         ⎜{0, 0, 0}        9    ⎟
         ⎜{0, 2}           12   ⎟
         ⎜{1, 1}           12   ⎟
         ⎜{4}              12   ⎟
         ⎜{0, 0, 0, 0}     38   ⎟
         ⎜{0, 0, 2}        99   ⎟
         ⎜{0, 1, 1}        125  ⎟
         ⎜{0, 4}           126  ⎟
         ⎜{1, 3}           138  ⎟
         ⎜{2, 2}           86   ⎟
         ⎜{6}              105  ⎟
         ⎜{0, 0, 0, 0, 0}  204  ⎟
         ⎜{0, 0, 0, 2}     1020 ⎟
         ⎜{0, 0, 1, 1}     1749 ⎟
         ⎜{0, 0, 4}        1473 ⎟
         ⎜{0, 1, 3}        3099 ⎟
         ⎜{0, 2, 2}        1622 ⎟
         ⎜{1, 1, 2}        1617 ⎟
         ⎜{0, 6}           1665 ⎟
         ⎜{1, 5}           1770 ⎟
         ⎜{2, 4}           1746 ⎟
         ⎜{3, 3}           962  ⎟
         ⎝{8}              1155 ⎠

In[57]:= MaxDualIndex /@ InvarDualCases[]

Out[57]= {1, 4, 3, 27, 58, 36, 32, 232, 967, 1047, 876, 920, 478, 435, 2582}

In[58]:= Array[MaxDualIndex, {5}]

Out[58]= {1, 4, 27, 232, 2582}
```

Within each case, the canonical invariants are labeled `(Dual)RInv[metric][case, index]`, where `index` is an integer ranging from 1 to `Max(Dual)Index[case]`. Non dual invariants are output as $I_{\text{case, index}}$, where case, for brevity and clarity, is not the list $\{\lambda_1, ..., \lambda_n\}$, but simply the concatenation of its elements $\lambda_1 \lambda_2 ... \lambda_n$. Notice that there is no confusion because the λ_i are always single digit integers, except for the case {10}, but even then there is no problem, because we always write the λ_i in increasing order. Dual invariants are represented by $D_{\text{case, index}}$.

Examples:

```
In[59]:= RInv[metric][{10}, 100] + RInv[metric][{0, 1, 3}, 1]

Out[59]= I10,100 + I013,1
```

```
In[60]:= RInv[metric][{0, 1, 2}, 12] - DualRInv[metric][{0, 0, 0}, 1]
```

Out[60]= $-D_{000,1} + I_{012,12}$

As we apply more symmetries, the lists of independent (Dual)RInvs get smaller. We can produce the independent invariant at each simplification step with the command `(Dual)RInvs[metric][step, case]`. The first step produces a list of length `MaxIndex[case]`

```
In[61]:= RInvs[metric][1, {2, 2}]
```

Out[61]= $\{I_{22,1}, I_{22,2}, I_{22,3}, I_{22,4}, I_{22,5}, I_{22,6}, I_{22,7}, I_{22,8}, I_{22,9}, I_{22,10}, I_{22,11}, I_{22,12}, I_{22,13}, I_{22,14}, I_{22,15}, I_{22,16}, I_{22,17}, I_{22,18}, I_{22,19}, I_{22,20}, I_{22,21}, I_{22,22}, I_{22,23}, I_{22,24}, I_{22,25}, I_{22,26}, I_{22,27}, I_{22,28}, I_{22,29}, I_{22,30}, I_{22,31}, I_{22,32}, I_{22,33}, I_{22,34}, I_{22,35}, I_{22,36}, I_{22,37}, I_{22,38}, I_{22,39}, I_{22,40}, I_{22,41}, I_{22,42}, I_{22,43}, I_{22,44}, I_{22,45}, I_{22,46}, I_{22,47}, I_{22,48}, I_{22,49}, I_{22,50}, I_{22,51}, I_{22,52}, I_{22,53}, I_{22,54}, I_{22,55}, I_{22,56}, I_{22,57}, I_{22,58}, I_{22,59}, I_{22,60}, I_{22,61}, I_{22,62}, I_{22,63}, I_{22,64}, I_{22,65}, I_{22,66}, I_{22,67}, I_{22,68}, I_{22,69}, I_{22,70}, I_{22,71}, I_{22,72}, I_{22,73}, I_{22,74}, I_{22,75}, I_{22,76}, I_{22,77}, I_{22,78}, I_{22,79}, I_{22,80}, I_{22,81}, I_{22,82}, I_{22,83}, I_{22,84}, I_{22,85}, I_{22,86}\}$

```
In[62]:= RInv[metric][{2}, #] & /@ Range[MaxIndex[{2, 2}]]
```

Out[62]= $\{I_{2,1}, I_{2,2}, I_{2,3}, I_{2,4}, I_{2,5}, I_{2,6}, I_{2,7}, I_{2,8}, I_{2,9}, I_{2,10}, I_{2,11}, I_{2,12}, I_{2,13}, I_{2,14}, I_{2,15}, I_{2,16}, I_{2,17}, I_{2,18}, I_{2,19}, I_{2,20}, I_{2,21}, I_{2,22}, I_{2,23}, I_{2,24}, I_{2,25}, I_{2,26}, I_{2,27}, I_{2,28}, I_{2,29}, I_{2,30}, I_{2,31}, I_{2,32}, I_{2,33}, I_{2,34}, I_{2,35}, I_{2,36}, I_{2,37}, I_{2,38}, I_{2,39}, I_{2,40}, I_{2,41}, I_{2,42}, I_{2,43}, I_{2,44}, I_{2,45}, I_{2,46}, I_{2,47}, I_{2,48}, I_{2,49}, I_{2,50}, I_{2,51}, I_{2,52}, I_{2,53}, I_{2,54}, I_{2,55}, I_{2,56}, I_{2,57}, I_{2,58}, I_{2,59}, I_{2,60}, I_{2,61}, I_{2,62}, I_{2,63}, I_{2,64}, I_{2,65}, I_{2,66}, I_{2,67}, I_{2,68}, I_{2,69}, I_{2,70}, I_{2,71}, I_{2,72}, I_{2,73}, I_{2,74}, I_{2,75}, I_{2,76}, I_{2,77}, I_{2,78}, I_{2,79}, I_{2,80}, I_{2,81}, I_{2,82}, I_{2,83}, I_{2,84}, I_{2,85}, I_{2,86}\}$

After applying the cyclic identity we get

```
In[63]:= RInvs[metric][2, {2, 2}]

           Reading InvRules for step 2 and case {2, 2}
```

Out[63]= $\{I_{22,1}, I_{22,2}, I_{22,3}, I_{22,4}, I_{22,5}, I_{22,6}, I_{22,7}, I_{22,8}, I_{22,9}, I_{22,10}, I_{22,11}, I_{22,12}, I_{22,13}, I_{22,14}, I_{22,15}, I_{22,16}, I_{22,17}, I_{22,18}, I_{22,19}, I_{22,20}, I_{22,21}, I_{22,22}, I_{22,23}, I_{22,24}, I_{22,26}, I_{22,27}, I_{22,28}, I_{22,29}, I_{22,31}, I_{22,32}, I_{22,34}, I_{22,35}, I_{22,36}, I_{22,37}, I_{22,38}, I_{22,39}, I_{22,40}, I_{22,41}, I_{22,44}, I_{22,46}, I_{22,48}, I_{22,50}, I_{22,52}, I_{22,54}, I_{22,55}, I_{22,56}, I_{22,57}, I_{22,58}, I_{22,59}, I_{22,67}, I_{22,68}, I_{22,69}, I_{22,73}, I_{22,74}, I_{22,75}, I_{22,76}, I_{22,77}, I_{22,78}, I_{22,85}\}$

After the Bianchi identity

```
In[64]:= RInvs[metric][3, {2, 2}]

           Reading InvRules for step 3 and case {2, 2}
```

Out[64]= $\{I_{22,1}, I_{22,2}, I_{22,3}, I_{22,4}, I_{22,6}, I_{22,8}, I_{22,10}, I_{22,15}, I_{22,16}, I_{22,17}, I_{22,18}, I_{22,19}, I_{22,20}, I_{22,21}, I_{22,22}, I_{22,27}, I_{22,34}, I_{22,38}, I_{22,44}, I_{22,67}, I_{22,73}, I_{22,74}, I_{22,77}\}$

After reordering derivatives

```
In[65]:= RInvs[metric][4, {2, 2}]

           Reading InvRules for step 4 and case {2, 2}
```

Out[65]= $\{I_{22,1}, I_{22,3}, I_{22,8}, I_{22,17}, I_{22,19}, I_{22,22}, I_{22,73}\}$

After the dimensionally dependent identities

```
In[66]:= RInvs[metric][5, {2, 2}]

          Reading InvRules for step 5, case {2, 2} and dimension 4
```

Out[66]= $\{I_{22,1}, I_{22,3}, I_{22,8}, I_{22,17}, I_{22,19}, I_{22,22}, I_{22,73}\}$

In this case there are no signature dependent relations (= relations arising from products of dual invariants)

```
In[67]:= RInvs[metric][6, {2, 2}]

          Reading InvRules for step 6, case {2, 2} and dimension 4
```

Out[67]= $\{I_{22,1}, I_{22,3}, I_{22,8}, I_{22,17}, I_{22,19}, I_{22,22}, I_{22,73}\}$

The following table reproduces the one in the paper listing the numbers of invariants after each step for all cases up to order 10

```
In[68]:= Table[Length[RInvs[step, InvarCases[][[case]]]], {step, 6}, {case, 26}];

          Reading InvRules for step 2 and case {0}
          Reading InvRules for step 2 and case {0, 0}
          Reading InvRules for step 2 and case {2}
          Reading InvRules for step 2 and case {0, 2}
          Reading InvRules for step 2 and case {1, 1}
          Reading InvRules for step 2 and case {4}
          Reading InvRules for step 2 and case {0, 0, 0, 0}
          Reading InvRules for step 2 and case {0, 0, 2}
          Reading InvRules for step 2 and case {0, 1, 1}
          Reading InvRules for step 2 and case {1, 3}
          Reading InvRules for step 2 and case {6}
          Reading InvRules for step 2 and case {0, 0, 0, 0, 0}
          Reading InvRules for step 2 and case {0, 0, 0, 2}
          Reading InvRules for step 2 and case {0, 0, 1, 1}
          Reading InvRules for step 2 and case {0, 0, 4}
          Reading InvRules for step 2 and case {0, 1, 3}
          Reading InvRules for step 2 and case {0, 2, 2}
          Reading InvRules for step 2 and case {1, 1, 2}
          Reading InvRules for step 2 and case {0, 6}
          Reading InvRules for step 2 and case {1, 5}
          Reading InvRules for step 2 and case {2, 4}
          Reading InvRules for step 2 and case {3, 3}
          Reading InvRules for step 2 and case {8}
          Reading InvRules for step 3 and case {0}
          Reading InvRules for step 3 and case {0, 0}
          Reading InvRules for step 3 and case {2}
          Reading InvRules for step 3 and case {0, 2}
          Reading InvRules for step 3 and case {1, 1}
          Reading InvRules for step 3 and case {4}
          Reading InvRules for step 3 and case {0, 0, 0, 0}
          Reading InvRules for step 3 and case {0, 0, 2}
          Reading InvRules for step 3 and case {0, 1, 1}
          Reading InvRules for step 3 and case {1, 3}
          Reading InvRules for step 3 and case {6}
          Reading InvRules for step 3 and case {0, 0, 0, 0, 0}
          Reading InvRules for step 3 and case {0, 0, 0, 2}
          Reading InvRules for step 3 and case {0, 0, 1, 1}
          Reading InvRules for step 3 and case {0, 0, 4}
          Reading InvRules for step 3 and case {0, 1, 3}
          Reading InvRules for step 3 and case {0, 2, 2}
          Reading InvRules for step 3 and case {1, 1, 2}
          Reading InvRules for step 3 and case {0, 6}
          Reading InvRules for step 3 and case {1, 5}
          Reading InvRules for step 3 and case {2, 4}
          Reading InvRules for step 3 and case {3, 3}
          Reading InvRules for step 3 and case {8}
          Reading InvRules for step 4 and case {0}
          Reading InvRules for step 4 and case {0, 0}
          Reading InvRules for step 4 and case {2}
          Reading InvRules for step 4 and case {0, 2}
          Reading InvRules for step 4 and case {1, 1}
          Reading InvRules for step 4 and case {4}
          Reading InvRules for step 4 and case {0, 0, 0, 0}
          Reading InvRules for step 4 and case {0, 0, 2}
          Reading InvRules for step 4 and case {0, 1, 1}
          Reading InvRules for step 4 and case {1, 3}
          Reading InvRules for step 4 and case {6}
          Reading InvRules for step 4 and case {0, 0, 0, 0, 0}
          Reading InvRules for step 4 and case {0, 0, 0, 2}
          Reading InvRules for step 4 and case {0, 0, 1, 1}
          Reading InvRules for step 4 and case {0, 0, 4}
          Reading InvRules for step 4 and case {0, 1, 3}
          Reading InvRules for step 4 and case {0, 2, 2}
          Reading InvRules for step 4 and case {1, 1, 2}
          Reading InvRules for step 4 and case {0, 6}
          Reading InvRules for step 4 and case {1, 5}
          Reading InvRules for step 4 and case {2, 4}
          Reading InvRules for step 4 and case {3, 3}
          Reading InvRules for step 4 and case {8}
          Reading InvRules for step 5, case {0} and dimension 4
          Reading InvRules for step 5, case {0, 0} and dimension 4
          Reading InvRules for step 5, case {2} and dimension 4
          Reading InvRules for step 5, case {0, 2} and dimension 4
          Reading InvRules for step 5, case {1, 1} and dimension 4
          Reading InvRules for step 5, case {4} and dimension 4
          Reading InvRules for step 5, case {1, 3} and dimension 4
          Reading InvRules for step 5, case {6} and dimension 4
          Reading InvRules for step 5, case {0, 0, 0, 0, 0} and dimension 4
          Reading InvRules for step 5, case {0, 0, 0, 2} and dimension 4
          Reading InvRules for step 5, case {0, 0, 1, 1} and dimension 4
          Reading InvRules for step 5, case {0, 0, 4} and dimension 4
          Reading InvRules for step 5, case {0, 1, 3} and dimension 4
          Reading InvRules for step 5, case {0, 2, 2} and dimension 4
          Reading InvRules for step 5, case {1, 1, 2} and dimension 4
          Reading InvRules for step 5, case {0, 6} and dimension 4
          Reading InvRules for step 5, case {1, 5} and dimension 4
          Reading InvRules for step 5, case {2, 4} and dimension 4
          Reading InvRules for step 5, case {3, 3} and dimension 4
          Reading InvRules for step 5, case {8} and dimension 4
          Reading InvRules for step 6, case {4} and dimension 4
          Reading InvRules for step 6, case {1, 3} and dimension 4
          Reading InvRules for step 6, case {6} and dimension 4
          Reading InvRules for step 6, case {0, 0, 0, 0, 0} and dimension 4
          Reading InvRules for step 6, case {0, 0, 0, 2} and dimension 4
          Reading InvRules for step 6, case {0, 0, 1, 1} and dimension 4
          Reading InvRules for step 6, case {0, 0, 4} and dimension 4
          Reading InvRules for step 6, case {0, 1, 3} and dimension 4
          Reading InvRules for step 6, case {0, 2, 2} and dimension 4
          Reading InvRules for step 6, case {1, 1, 2} and dimension 4
          Reading InvRules for step 6, case {0, 6} and dimension 4
          Reading InvRules for step 6, case {1, 5} and dimension 4
          Reading InvRules for step 6, case {2, 4} and dimension 4
          Reading InvRules for step 6, case {3, 3} and dimension 4
          Reading InvRules for step 6, case {8} and dimension 4
```

```
In[69]:= Transpose[Join[{InvarCases[][[Range[26]]]}, %]] // MatrixForm
```

```
Out[69]//MatrixForm=
{0}              1     1     1    1    1   1
{0, 0}           3     2     2    2    2   2
{2}              2     2     1    1    1   1
{0, 0, 0}        9     5     5    5    3   3
{0, 2}           12    9     5    3    3   3
{1, 1}           12    9     4    4    4   4
{4}              12    11    6    1    1   1
{0, 0, 0, 0}     38    15    15   15   4   3
{0, 0, 2}        99    48    27   15   10  10
{0, 1, 1}        125   63    23   23   17  17
{0, 4}           126   84    47   3    3   3
{1, 3}           138   95    32   5    5   5
{2, 2}           86    59    23   7    7   7
{6}              105   90    50   1    1   1
{0, 0, 0, 0, 0}  204   54    54   54   5   3
{0, 0, 0, 2}     1020  313   175  79   26  25
{0, 0, 1, 1}     1749  564   194  194  76  74
{0, 0, 4}        1473  648   361  17   12  12
{0, 1, 3}        3099  1387  442  53   42  42
{0, 2, 2}        1622  727   244  46   34  34
{1, 1, 2}        1617  741   143  67   52  52
{0, 6}           1665  1025  570  3    3   3
{1, 5}           1770  1115  362  5    5   5
{2, 4}           1746  1093  356  9    9   9
{3, 3}           962   612   211  9    9   9
{8}              1155  945   525  1    1   1
```

The simplified notation `RInvs[metric][step, n]` is automatically translated into `RInvs[metric][step, {0, .n., 0}]`, for backwards compatibility with Invar 1. For instance, these are all the algebraic invariants after step 3 with degree 6 (case {0,0,0,0,0,0}):

```
In[70]:= RInvs[metric][3, 6]

            Reading InvRules for step 2 and case {0, 0, 0, 0, 0, 0}

            Reading InvRules for step 3 and case {0, 0, 0, 0, 0, 0}
```

Out[70]=

$$\{I_{000000,1}, I_{000000,2}, I_{000000,3}, I_{000000,4}, I_{000000,6}, I_{000000,8}, I_{000000,9}, I_{000000,11}, I_{000000,13}, I_{000000,14}, I_{000000,16}, I_{000000,17}, I_{000000,18}, I_{000000,19}, I_{000000,22}, I_{000000,23}, I_{000000,25}, I_{000000,27}, I_{000000,32}, I_{000000,34}, I_{000000,36}, I_{000000,38}, I_{000000,40}, I_{000000,45}, I_{000000,47}, I_{000000,49}, I_{000000,50}, I_{000000,52}, I_{000000,57}, I_{000000,60}, I_{000000,65}, I_{000000,66}, I_{000000,69}, I_{000000,71}, I_{000000,74}, I_{000000,76}, I_{000000,77}, I_{000000,84}, I_{000000,86}, I_{000000,87}, I_{000000,88}, I_{000000,91}, I_{000000,94}, I_{000000,95}, I_{000000,97}, I_{000000,101}, I_{000000,103}, I_{000000,108}, I_{000000,110}, I_{000000,112}, I_{000000,114}, I_{000000,117}, I_{000000,119}, I_{000000,136}, I_{000000,140}, I_{000000,143}, I_{000000,147}, I_{000000,149}, I_{000000,151}, I_{000000,156}, I_{000000,165}, I_{000000,167}, I_{000000,174}, I_{000000,176}, I_{000000,178}, I_{000000,180}, I_{000000,183}, I_{000000,187}, I_{000000,189}, I_{000000,191}, I_{000000,196}, I_{000000,208}, I_{000000,213}, I_{000000,216}, I_{000000,221}, I_{000000,223}, I_{000000,229}, I_{000000,231}, I_{000000,233}, I_{000000,235}, I_{000000,240}, I_{000000,242}, I_{000000,244}, I_{000000,245}, I_{000000,247}, I_{000000,252}, I_{000000,255}, I_{000000,260}, I_{000000,261}, I_{000000,264}, I_{000000,266}, I_{000000,268}, I_{000000,270}, I_{000000,275}, I_{000000,292}, I_{000000,296}, I_{000000,298}, I_{000000,304}, I_{000000,309}, I_{000000,312}, I_{000000,331}, I_{000000,334}, I_{000000,342}, I_{000000,344}, I_{000000,346}, I_{000000,348}, I_{000000,349}, I_{000000,353}, I_{000000,359}, I_{000000,362}, I_{000000,372}, I_{000000,374}, I_{000000,379}, I_{000000,381}, I_{000000,383}, I_{000000,385}, I_{000000,393}, I_{000000,395}, I_{000000,400}, I_{000000,402}, I_{000000,404}, I_{000000,406}, I_{000000,409}, I_{000000,411}, I_{000000,428}, I_{000000,432}, I_{000000,435}, I_{000000,439}, I_{000000,441}, I_{000000,443}, I_{000000,448}, I_{000000,457}, I_{000000,459}, I_{000000,464}, I_{000000,466}, I_{000000,468}, I_{000000,470}, I_{000000,474}, I_{000000,478}, I_{000000,480}, I_{000000,484}, I_{000000,492}, I_{000000,497}, I_{000000,499}, I_{000000,503}, I_{000000,505}, I_{000000,510}, I_{000000,533}, I_{000000,535}, I_{000000,539}, I_{000000,544}, I_{000000,561}, I_{000000,569}, I_{000000,625}, I_{000000,627}, I_{000000,629}, I_{000000,633}, I_{000000,650}, I_{000000,655}, I_{000000,658}, I_{000000,665}, I_{000000,670}, I_{000000,672}, I_{000000,701}, I_{000000,703}, I_{000000,708}, I_{000000,712}, I_{000000,714}, I_{000000,716}, I_{000000,719}, I_{000000,725}, I_{000000,732}, I_{000000,738}, I_{000000,740}, I_{000000,742}, I_{000000,749}, I_{000000,751}, I_{000000,754}, I_{000000,765}, I_{000000,767}, I_{000000,772}, I_{000000,782}, I_{000000,791}, I_{000000,797}, I_{000000,802}, I_{000000,828}, I_{000000,829}, I_{000000,839}, I_{000000,864}, I_{000000,869}, I_{000000,898}, I_{000000,928}, I_{000000,930}, I_{000000,935}, I_{000000,937}, I_{000000,939}, I_{000000,941}, I_{000000,944}, I_{000000,946}, I_{000000,963}, I_{000000,967}, I_{000000,970}, I_{000000,974}, I_{000000,976}, I_{000000,978}, I_{000000,983}, I_{000000,993}, I_{000000,998}, I_{000000,1001}, I_{000000,1003}, I_{000000,1006}, I_{000000,1009}, I_{000000,1011}, I_{000000,1015}, I_{000000,1023}, I_{000000,1028}, I_{000000,1030}, I_{000000,1045}, I_{000000,1048}, I_{000000,1053}, I_{000000,1058}, I_{000000,1068}, I_{000000,1073}, I_{000000,1076}, I_{000000,1081}, I_{000000,1083}, I_{000000,1131}, I_{000000,1142}, I_{000000,1146}, I_{000000,1148}, I_{000000,1153}, I_{000000,1158}, I_{000000,1161}, I_{000000,1179}, I_{000000,1182}, I_{000000,1190}, I_{000000,1192}, I_{000000,1194}, I_{000000,1201}, I_{000000,1203}, I_{000000,1204}, I_{000000,1213}, I_{000000,1219}, I_{000000,1221}, I_{000000,1223}, I_{000000,1478}, I_{000000,1480}, I_{000000,1482}, I_{000000,1486}, I_{000000,1488}, I_{000000,1502}, I_{000000,1504}, I_{000000,1508}, I_{000000,1510}, I_{000000,1519}, I_{000000,1523}, I_{000000,1530}, I_{000000,1541}, I_{000000,1542}, I_{000000,1548}, I_{000000,1561}, I_{000000,1563}, I_{000000,1572}, I_{000000,1576}, I_{000000,1577}, I_{000000,1579}, I_{000000,1582}, I_{000000,1586}, I_{000000,1588}, I_{000000,1596}\}$$

Similar shorthands are available for other commands, such as `MaxIndex` or `RInv`

```
In[71]:= {MaxIndex[7], MaxIndex[{0, 0, 0, 0, 0, 0, 0}]}

Out[71]= {16532, 16532}
```

```
In[72]:= RInv[metric][4, 1]
```

Out[72]= $I_{0000,1}$

```
In[73]:= DualRInvs[2, 4]

           Reading DualInvRules for step 2 and case {0, 0, 0, 0}
```

Out[73]= $\{D_{0000,2}, D_{0000,7}, D_{0000,10}, D_{0000,13}, D_{0000,17}, D_{0000,18}, D_{0000,23}, D_{0000,26}, D_{0000,27}, D_{0000,29}, D_{0000,35}, D_{0000,38}, D_{0000,43}, D_{0000,44}, D_{0000,51}, D_{0000,60}, D_{0000,68}, D_{0000,70}, D_{0000,80}, D_{0000,82}, D_{0000,85}, D_{0000,94}, D_{0000,96}, D_{0000,100}, D_{0000,113}, D_{0000,116}, D_{0000,121}, D_{0000,122}, D_{0000,127}, D_{0000,128}, D_{0000,131}, D_{0000,138}, D_{0000,145}, D_{0000,148}, D_{0000,149}, D_{0000,201}, D_{0000,202}, D_{0000,210}, D_{0000,215}, D_{0000,227}\}$

If we restrict ourselves to algebraic invariants we can give a basis of only 25 invariants (notice that there is no step 6 for dual invariants)

```
In[74]:= Flatten[Join[RInvs[metric][6, #] & /@ Range[7], DualRInvs[metric][5, #] & /@ Range[5]]]

           Reading InvRules for step 4 and case {0, 0, 0, 0, 0, 0}

           Reading InvRules for step 5, case {0, 0, 0, 0, 0, 0} and dimension 4

           Reading InvRules for step 6, case {0, 0, 0, 0, 0, 0} and dimension 4

           Reading InvRules for step 2 and case {0, 0, 0, 0, 0, 0, 0}

           Reading InvRules for step 3 and case {0, 0, 0, 0, 0, 0, 0}

           Reading InvRules for step 4 and case {0, 0, 0, 0, 0, 0, 0}

           Reading InvRules for step 5, case {0, 0, 0, 0, 0, 0, 0} and dimension 4

           Reading InvRules for step 6, case {0, 0, 0, 0, 0, 0, 0} and dimension 4

           Reading DualInvRules for step 2 and case {0}

           Reading DualInvRules for step 3 and case {0}

           Reading DualInvRules for step 4 and case {0}

           Reading DualInvRules for step 5 and case {0}

           Reading DualInvRules for step 2 and case {0, 0, 0}

           Reading DualInvRules for step 3 and case {0, 0, 0}

           Reading DualInvRules for step 4 and case {0, 0, 0}

           Reading DualInvRules for step 5 and case {0, 0, 0}

           Reading DualInvRules for step 3 and case {0, 0, 0, 0}

           Reading DualInvRules for step 4 and case {0, 0, 0, 0}

           Reading DualInvRules for step 5 and case {0, 0, 0, 0}
```

Out[74]= $\{I_{0,1}, I_{00,1}, I_{00,2}, I_{000,1}, I_{000,2}, I_{000,5}, I_{0000,1}, I_{0000,5}, I_{0000,7}, I_{00000,2}, I_{00000,8}, I_{00000,33}, I_{000000,6}, I_{000000,8}, I_{000000,47}, I_{000000,242}, I_{0000000,14}, I_{0000000,55}, I_{0000000,391}, D_{00,2}, D_{000,2}, D_{000,13}, D_{0000,7}, D_{00000,2}, D_{00000,76}\}$

```
In[75]:=  Transpose[{%, InvToRiemann[%, False]}] // TableForm
```

```
Reading InvRules for step 1 and case {0, 0, 0, 0, 0}
Reading InvRules for step 1 and case {0, 0, 0, 0, 0, 0}
Reading InvRules for step 1 and case {0, 0, 0, 0, 0, 0, 0}
Reading DualInvRules for step 1 and case {0, 0, 0}
Reading DualInvRules for step 1 and case {0, 0, 0, 0}
```

Out[75]//TableForm=

$I_{0,1}$ Scalar$[R^{ab}{}_{ab}]$

$I_{00,1}$ Scalar$[R^{ab}{}_{a}{}^{c}\, R_{b}{}^{d}{}_{cd}]$

$I_{00,2}$ Scalar$[R_{abcd}\, R^{abcd}]$

$I_{000,1}$ Scalar$[R^{ab}{}_{a}{}^{c}\, R_{b}{}^{d}{}_{d}{}^{e}\, R_{c}{}^{f}{}_{ef}]$

$I_{000,2}$ Scalar$[R^{ab}{}_{a}{}^{c}\, R_{b}{}^{d}{}_{c}{}^{e}\, R_{d}{}^{f}{}_{ef}]$

$I_{000,5}$ Scalar$[R_{ab}{}^{ef}\, R^{abcd}\, R_{cdef}]$

$I_{0000,1}$ Scalar$[R^{ab}{}_{a}{}^{c}\, R_{b}{}^{d}{}_{d}{}^{e}\, R_{c}{}^{f}{}_{f}{}^{g}\, R_{e}{}^{h}{}_{gh}]$

$I_{0000,5}$ Scalar$[R^{ab}{}_{a}{}^{c}\, R_{b}{}^{def}\, R_{c}{}^{g}{}_{ef}\, R_{d}{}^{h}{}_{gh}]$

$I_{0000,7}$ Scalar$[R^{ab}{}_{a}{}^{c}\, R_{b}{}^{d}{}_{c}{}^{e}\, R_{d}{}^{f}{}_{e}{}^{g}\, R_{f}{}^{h}{}_{gh}]$

$I_{00000,2}$ Scalar$[R^{ab}{}_{a}{}^{c}\, R_{b}{}^{d}{}_{c}{}^{e}\, R_{d}{}^{f}{}_{f}{}^{g}\, R_{e}{}^{h}{}_{h}{}^{i}\, R_{g}{}^{j}{}_{ij}]$

$I_{00000,8}$ Scalar$[R^{ab}{}_{a}{}^{c}\, R_{b}{}^{d}{}_{c}{}^{e}\, R_{d}{}^{f}{}_{e}{}^{g}\, R_{f}{}^{h}{}_{h}{}^{i}\, R_{g}{}^{j}{}_{ij}]$

$I_{00000,33}$ Scalar$[R^{ab}{}_{a}{}^{c}\, R_{b}{}^{d}{}_{c}{}^{e}\, R_{d}{}^{fgh}\, R_{e}{}^{i}{}_{gh}\, R_{f}{}^{j}{}_{ij}]$

$I_{000000,6}$ Scalar$[R^{ab}{}_{a}{}^{c}\, R_{b}{}^{d}{}_{d}{}^{e}\, R_{c}{}^{f}{}_{f}{}^{g}\, R_{e}{}^{hij}\, R_{g}{}^{k}{}_{ij}\, R_{h}{}^{l}{}_{kl}]$

$I_{000000,8}$ Scalar$[R^{ab}{}_{a}{}^{c}\, R_{b}{}^{d}{}_{c}{}^{e}\, R_{d}{}^{f}{}_{e}{}^{g}\, R_{f}{}^{h}{}_{h}{}^{i}\, R_{g}{}^{j}{}_{j}{}^{k}\, R_{i}{}^{l}{}_{kl}]$

$I_{000000,47}$ Scalar$[R^{ab}{}_{a}{}^{c}\, R_{b}{}^{d}{}_{d}{}^{e}\, R_{c}{}^{f}{}_{e}{}^{g}\, R_{f}{}^{hij}\, R_{g}{}^{k}{}_{ij}\, R_{h}{}^{l}{}_{kl}]$

$I_{000000,242}$ Scalar$[R^{ab}{}_{a}{}^{c}\, R_{b}{}^{d}{}_{c}{}^{e}\, R_{d}{}^{fgh}\, R_{e}{}^{i}{}_{gh}\, R_{f}{}^{j}{}_{i}{}^{k}\, R_{j}{}^{l}{}_{kl}]$

$I_{0000000,14}$ Scalar$[R^{ab}{}_{a}{}^{c}\, R_{b}{}^{d}{}_{d}{}^{e}\, R_{c}{}^{f}{}_{e}{}^{g}\, R_{f}{}^{h}{}_{g}{}^{i}\, R_{h}{}^{j}{}_{j}{}^{k}\, R_{i}{}^{l}{}_{l}{}^{m}\, R_{k}{}^{n}{}_{mn}]$

$I_{0000000,55}$ Scalar$[R^{ab}{}_{a}{}^{c}\, R_{b}{}^{d}{}_{d}{}^{e}\, R_{c}{}^{f}{}_{f}{}^{g}\, R_{e}{}^{h}{}_{g}{}^{i}\, R_{h}{}^{jkl}\, R_{i}{}^{m}{}_{kl}\, R_{j}{}^{n}{}_{mn}]$

$I_{0000000,391}$ Scalar$[R^{ab}{}_{a}{}^{c}\, R_{b}{}^{d}{}_{c}{}^{e}\, R_{d}{}^{fgh}\, R_{e}{}^{i}{}_{gh}\, R_{f}{}^{j}{}_{i}{}^{k}\, R_{j}{}^{l}{}_{l}{}^{m}\, R_{k}{}^{n}{}_{mn}]$

$D_{00,2}$ Scalar$[\epsilon_{cdef}\, R_{ab}{}^{ef}\, R^{abcd}]$

$D_{000,2}$ Scalar$[\epsilon_{cefh}\, R^{ab}{}_{a}{}^{c}\, R_{b}{}^{def}\, R_{d}{}^{g}{}_{g}{}^{h}]$

$D_{000,13}$ Scalar$[\epsilon_{efgh}\, R_{ab}{}^{ef}\, R^{abcd}\, R_{cd}{}^{gh}]$

$D_{0000,7}$ Scalar$[\epsilon_{eghj}\, R^{ab}{}_{a}{}^{c}\, R_{b}{}^{d}{}_{c}{}^{e}\, R_{d}{}^{fgh}\, R_{f}{}^{i}{}_{i}{}^{j}]$

$D_{00000,2}$ Scalar$[\epsilon_{gijl}\, R^{ab}{}_{a}{}^{c}\, R_{b}{}^{d}{}_{d}{}^{e}\, R_{c}{}^{f}{}_{f}{}^{g}\, R_{e}{}^{hij}\, R_{h}{}^{k}{}_{k}{}^{l}]$

$D_{00000,76}$ Scalar$[\epsilon_{gijl}\, R^{ab}{}_{a}{}^{c}\, R_{b}{}^{d}{}_{c}{}^{e}\, R_{d}{}^{f}{}_{e}{}^{g}\, R_{f}{}^{hij}\, R_{h}{}^{k}{}_{k}{}^{l}]$

■ 4. Inv –> Permutation –> Riemann

`InvToPerm`	Conversion form invariants (RInv) to permutations (RPerm)
`PermToRiemann`	Conversion from permutations (RPerm) to Riemann tensor expressions
`InvToRiemann`	Direct conversion from invariants (RInv) to Riemann tensor expressions

From invariants to Riemann expressions.

4.1. InvToPerm

Given an invariant, it is possible to know the permutation it represents using `InvToPerm`. The result is returned using the head `RPerm`, which stands for permutation of Riemann invariant.

This is a typical example of a canonical invariant:

```
In[76]:=  rinv = 7 RInv[metric][{0, 4}, 100]

Out[76]=  7 I04,100

In[77]:=  dualrinv = 3 DualRInv[metric][{0, 0, 2}, 1]

Out[77]=  3 D002,1
```

And this is the permutation it represents. It is given in cyclic form. The first list identifies the case and whether it contains an epsilon ({case, 1}) or not ({case, 0})

```
In[78]:=  rperm = InvToPerm[rinv]

Out[78]=  7 RPerm[metric][{{0, 4}, 0}, Cycles[{2, 3, 5}, {4, 7, 9, 10, 6}, {8, 11}]]

In[79]:=  dperm = InvToPerm[dualrinv]

            Reading DualInvRules for step 1 and case {0, 0, 2}

Out[79]=  3 RPerm[metric][{{0, 0, 2}, 1},
            Cycles[{2, 3}, {4, 5}, {6, 7, 8, 9}, {10, 11, 12, 13, 15}, {14, 17, 16}]]
```

The command also works with polynomials, giving one RPerm for each monomial

```
In[80]:=  InvToPerm[rinv + dualrinv]

Out[80]=  7 RPerm[metric][{{0, 4}, 0}, Cycles[{2, 3, 5}, {4, 7, 9, 10, 6}, {8, 11}]] +
           3 RPerm[metric][{{0, 0, 2}, 1},
             Cycles[{2, 3}, {4, 5}, {6, 7, 8, 9}, {10, 11, 12, 13, 15}, {14, 17, 16}]]
```

4.2. PermToRiemann

Any permutation of *N* indices can be converted into a Riemann monomial of the appropiate case with the function `PermToRiemann`. There is a second argument to specify whether we want the contracted Riemann tensors to be automatically transformed into Ricci or not.

The default value of the argument for conversion into Ricci is stored in the global variable

```
In[81]:=  $CurvatureRelations

Out[81]=  False

In[82]:=  rinv2 = RInv[metric][{0, 0}, 1]

Out[82]=  I00,1
```

```
In[83]:=  rperm2 = InvToPerm[rinv2]

Out[83]=  RPerm[metric][{{0, 0}, 0}, Cycles[{2, 3}, {4, 5}, {6, 7}]]
```

If we set that variable to True, we get a product of Ricci tensors:

```
In[84]:=  PermToRiemann[rperm2, True]
```

Out[84]= Scalar[$R_{ab} R^{ba}$]

If we set that argument to False or use the default, no Riccis appear

```
In[85]:=  PermToRiemann[rperm2, False]
```

Out[85]= Scalar[$R^{ab}{}_{a}{}^{c} R_{b}{}^{d}{}_{cd}$]

```
In[86]:=  PermToRiemann[rperm2]
```

Out[86]= Scalar[$R^{ab}{}_{a}{}^{c} R_{b}{}^{d}{}_{cd}$]

4.3. InvToRiemann

There is a simple function which combines the action of the previous two. It also has a second argument:

In a single line:

```
In[87]:=  InvToRiemann[RInv[metric][{0, 0, 0, 0, 0, 0, 0}, 1000]]
```

Out[87]= Scalar[$R^{ab}{}_{a}{}^{c} R_{b}{}^{def} R_{cd}{}^{gh} R_{eg}{}^{ij} R_{f}{}^{k}{}_{i}{}^{l} R_{h}{}^{m}{}_{lm} R_{j}{}^{n}{}_{kn}$]

It is possible to give a list of invariants:

```
In[88]:=  RInvs[metric][5, {0, 0, 2}]
```

Out[88]= {$I_{002,1}$, $I_{002,2}$, $I_{002,5}$, $I_{002,6}$, $I_{002,7}$, $I_{002,8}$, $I_{002,14}$, $I_{002,41}$, $I_{002,52}$, $I_{002,55}$}

```
In[89]:=  Map[InvToRiemann, %]
```

Out[89]= {Scalar[$R^{ab}{}_{a}{}^{c} R_{b}{}^{d}{}_{d}{}^{e} R^{fg}{}_{fg;c;e}$], Scalar[$R^{ab}{}_{a}{}^{c} R_{b}{}^{d}{}_{d}{}^{e} R_{c}{}^{f}{}_{ef}{}^{;g}{}_{;g}$],
Scalar[$R^{ab}{}_{a}{}^{c} R^{de}{}_{d}{}^{f} R_{b}{}^{g}{}_{cg;e;f}$], Scalar[$R^{ab}{}_{a}{}^{c} R^{de}{}_{d}{}^{f} R_{b}{}^{g}{}_{eg;c;f}$],
Scalar[$R^{ab}{}_{a}{}^{c} R_{b}{}^{d}{}_{c}{}^{e} R^{fg}{}_{fg;d;e}$], Scalar[$R^{ab}{}_{a}{}^{c} R_{b}{}^{d}{}_{c}{}^{e} R_{d}{}^{f}{}_{ef}{}^{;g}{}_{;g}$], Scalar[$R^{ab}{}_{a}{}^{c} R_{b}{}^{def} R_{c}{}^{g}{}_{eg;d;f}$],
Scalar[$R^{ab}{}_{a}{}^{c} R^{defg} R_{bdcf;e;g}$], Scalar[$R_{ab}{}^{ef} R^{abcd} R_{c}{}^{g}{}_{eg;d;f}$], Scalar[$R_{a}{}^{e}{}_{c}{}^{f} R^{abcd} R_{b}{}^{g}{}_{dg;e;f}$]}

```
In[90]:= Transpose[{%%, %}] // TableForm
```

Out[90]//TableForm=

$I_{002,1}$ $\quad$ $\text{Scalar}[R^{ab}{}_{a}{}^{c}\, R_{b}{}^{d}{}_{d}{}^{e}\, R^{fg}{}_{fg;c;e}]$

$I_{002,2}$ $\quad$ $\text{Scalar}[R^{ab}{}_{a}{}^{c}\, R_{b}{}^{d}{}_{d}{}^{e}\, R_{c}{}^{f}{}_{ef}{}^{;g}{}_{;g}]$

$I_{002,5}$ $\quad$ $\text{Scalar}[R^{ab}{}_{a}{}^{c}\, R^{de}{}_{d}{}^{f}\, R_{b}{}^{g}{}_{cg;e;f}]$

$I_{002,6}$ $\quad$ $\text{Scalar}[R^{ab}{}_{a}{}^{c}\, R^{de}{}_{d}{}^{f}\, R_{b}{}^{g}{}_{eg;c;f}]$

$I_{002,7}$ $\quad$ $\text{Scalar}[R^{ab}{}_{a}{}^{c}\, R_{b}{}^{d}{}_{c}{}^{e}\, R^{fg}{}_{fg;d;e}]$

$I_{002,8}$ $\quad$ $\text{Scalar}[R^{ab}{}_{a}{}^{c}\, R_{b}{}^{d}{}_{c}{}^{e}\, R_{d}{}^{f}{}_{ef}{}^{;g}{}_{;g}]$

$I_{002,14}$ $\quad$ $\text{Scalar}[R^{ab}{}_{a}{}^{c}\, R_{b}{}^{def}\, R_{c}{}^{g}{}_{eg;d;f}]$

$I_{002,41}$ $\quad$ $\text{Scalar}[R^{ab}{}_{a}{}^{c}\, R^{defg}\, R_{bdcf;e;g}]$

$I_{002,52}$ $\quad$ $\text{Scalar}[R_{ab}{}^{ef}\, R^{abcd}\, R_{c}{}^{g}{}_{eg;d;f}]$

$I_{002,55}$ $\quad$ $\text{Scalar}[R_{a}{}^{e}{}_{c}{}^{f}\, R^{abcd}\, R_{b}{}^{g}{}_{dg;e;f}]$

With contracted Riemanns substituted by Riccis,

```
In[91]:= RInvs[metric][5, {0, 0, 2}]
```

Out[91]= $\{I_{002,1}, I_{002,2}, I_{002,5}, I_{002,6}, I_{002,7}, I_{002,8}, I_{002,14}, I_{002,41}, I_{002,52}, I_{002,55}\}$

```
In[92]:= Map[InvToRiemann[#, True] &, %]
```

Out[92]= $\{-\text{Scalar}[R^{b}{}_{a}\, R_{c}{}^{a}\, R^{;c}{}_{;b}],\ -\text{Scalar}[R_{ca}\, R_{d}{}^{a}\, R^{dc;b}{}_{;b}],$
$\text{Scalar}[R^{a}{}_{d}\, R_{bc}\, R^{cb;d}{}_{;a}],\ \text{Scalar}[R^{a}{}_{d}\, R_{bc}\, R^{cd;b}{}_{;a}],\ \text{Scalar}[R^{ba}\, R_{a}{}^{c}{}_{b}{}^{d}\, R_{;c;d}],$
$\text{Scalar}[R^{ba}\, R_{a}{}^{c}{}_{b}{}^{d}\, R_{cd}{}^{;e}{}_{;e}],\ \text{Scalar}[R_{e}{}^{a}\, R_{a}{}^{bcd}\, R^{e}{}_{c;b;d}],\ \text{Scalar}[R_{ef}\, R^{abcd}\, R^{f}{}_{a}{}^{e}{}_{c;b;d}],$
$\text{Scalar}[R_{ab}{}^{ef}\, R^{abcd}\, R_{ce;d;f}],\ \text{Scalar}[R_{a}{}^{e}{}_{c}{}^{f}\, R^{abcd}\, R_{bd;e;f}]\}$

```
In[93]:= Transpose[{%%, %}] // TableForm
```

Out[93]//TableForm=

$I_{002,1}$ $\quad$ $-\text{Scalar}[R^{b}{}_{a}\, R_{c}{}^{a}\, R^{;c}{}_{;b}]$

$I_{002,2}$ $\quad$ $-\text{Scalar}[R_{ca}\, R_{d}{}^{a}\, R^{dc;b}{}_{;b}]$

$I_{002,5}$ $\quad$ $\text{Scalar}[R^{a}{}_{d}\, R_{bc}\, R^{cb;d}{}_{;a}]$

$I_{002,6}$ $\quad$ $\text{Scalar}[R^{a}{}_{d}\, R_{bc}\, R^{cd;b}{}_{;a}]$

$I_{002,7}$ $\quad$ $\text{Scalar}[R^{ba}\, R_{a}{}^{c}{}_{b}{}^{d}\, R_{;c;d}]$

$I_{002,8}$ $\quad$ $\text{Scalar}[R^{ba}\, R_{a}{}^{c}{}_{b}{}^{d}\, R_{cd}{}^{;e}{}_{;e}]$

$I_{002,14}$ $\quad$ $\text{Scalar}[R_{e}{}^{a}\, R_{a}{}^{bcd}\, R^{e}{}_{c;b;d}]$

$I_{002,41}$ $\quad$ $\text{Scalar}[R_{ef}\, R^{abcd}\, R^{f}{}_{a}{}^{e}{}_{c;b;d}]$

$I_{002,52}$ $\quad$ $\text{Scalar}[R_{ab}{}^{ef}\, R^{abcd}\, R_{ce;d;f}]$

$I_{002,55}$ $\quad$ $\text{Scalar}[R_{a}{}^{e}{}_{c}{}^{f}\, R^{abcd}\, R_{bd;e;f}]$

Note that we have got a minus sign. This is because the invariants are sorted with respect to their Riemann−only expression:

■ 5. Riemann −> Permutation −> Inv

`RiemannToPerm`	Conversion from Riemann tensor expressions into permutations (RPerm)
`PermToInv`	Strong Conversion from permutations (RPerm) into invariants (RInv)
`RiemannToInv`	Direct conversion from Riemann tensor expressions into invariants (RInv)

From Riemann expressions to invariants.

5.1. RiemannToPerm

The function `RiemannToPerm` converts all Riemann scalars of a given metric (or list of metrics) into their canonical permutations:

Suppose we start from this simple expression:

```
In[94]:= rexpr = RiemannCD[a, b, c, d] RiemannCD[-a, -b, -c, -d] +
           6 RiemannCD[e, f, -c, -d] RiemannCD[-a, -b, -e, -f] epsilonmetric[a, b, c, d]
```

Out[94]= $R_{abcd} R^{abcd} + 6 \epsilon^{abcd} R_{abef} R^{ef}{}_{cd}$

Then we can transform all terms into their *canonical* permutations,

```
In[95]:= rperm = RiemannToPerm[rexpr]

Out[95]= RPerm[metric][{{0, 0}, 0}, Cycles[{2, 3, 5}, {4, 7, 6}]] +
          6 RPerm[metric][{{0, 0}, 1}, Cycles[{2, 3, 5}, {4, 7, 9, 6}, {8, 11, 10}]]
```

that is, not the permutations which would regenerate the previous objects:

```
In[96]:= PermToRiemann[rperm]
```

Out[96]= Scalar$[R_{abcd} R^{abcd}]$ + 6 Scalar$[\epsilon_{cdef} R_{ab}{}^{ef} R^{abcd}]$

5.2. PermToInv

This function uses the stored tables of results to identify the index corresponding to a given canonical permutation.

Example:

```
In[97]:= PermToInv[rperm]
```

Out[97]= $6 D_{00,2} + I_{00,2}$

This function only works with canonical permutations:

```
In[98]:=  InvToPerm[RInv[metric][{0, 2}, 3]]

Out[98]=  RPerm[metric][{{0, 2}, 0}, Cycles[{2, 3}, {4, 5}, {6, 7, 8, 9}]]

In[99]:=  PermToInv[%]
```

Out[99]= $I_{02,3}$

We rewrite the cycles in a different form

```
In[100]:=
      RPerm[metric][{{0, 2}, 0}, Cycles[{3, 2}, {4, 5}, {6, 7, 8, 9}]]

Out[100]=
      RPerm[metric][{{0, 2}, 0}, Cycles[{3, 2}, {4, 5}, {6, 7, 8, 9}]]
```

but now PermToInv does not work, because the new permutation is not canonical

```
In[101]:=
      PermToInv[%]

Out[101]=
      Cycles[{3, 2}, {4, 5}, {6, 7, 8, 9}]
```

5.3. RiemannToInv

Combined function:

Simple example:

```
In[102]:=
      RiemannToInv[rexpr]
```

Out[102]=

$6\, D_{00,2} + I_{00,2}$

■ 6. Simplification

6.1. InvSimplify

Once we have converted any Riemann expression into invariants with heads `RInv` and `DualRInv`, then we can "simplify" the expression by using the multiterm symmetries described above (steps 2, 3 and 4 of the simplification process). We use quotes because in many cases a single monomial is expanded into a large polynomial, its canonical form in the basis we have chosen, and this can be hardly called simplification. However, if an expression is equivalent to zero, the function InvSimplify will find it.

Example:

```
In[103]:=
      rinv = RInv[metric][{0, 0, 0, 0, 0}, 100]
```

Out[103]=

$$I_{00000,100}$$

At level 1 nothing happens:

```
In[104]:=
      InvSimplify[rinv, 1]
```

Out[104]=

$$I_{00000,100}$$

At level 2 the cyclic identity is used. Note that the degree does not change, but the indexes of the invariants decrease:

```
In[105]:=
      InvSimplify[rinv, 2]
```

Out[105]=

$$-I_{00000,97} + I_{00000,99}$$

We skip levels 3 and 4 becasue there are no derivatives in this example. At level 5 we use dimensionally dependent identities for dimension 4:

```
In[106]:=
      InvSimplify[rinv, 5] // Expand
```

Out[106]=

$$-\frac{11 I_{0,1}^5}{96} + \frac{79}{96} I_{0,1}^3 I_{00,1} - \frac{1}{8} I_{0,1} I_{00,1}^2 - \frac{1}{192} I_{0,1}^3 I_{00,2} + \frac{5}{32} I_{0,1} I_{00,1} I_{00,2} - \frac{1}{64} I_{0,1} I_{00,2}^2 + \frac{5}{3} I_{0,1}^2 I_{000,1} + \frac{I_{00,1} I_{000,1}}{6} + \frac{I_{00,2} I_{000,1}}{6} + \frac{1}{4} I_{0,1}^2 I_{000,2} - \frac{I_{00,1} I_{000,2}}{2} - \frac{I_{00,2} I_{000,2}}{8} - \frac{1}{32} I_{0,1}^2 I_{000,5} + \frac{3 I_{0,1} I_{0000,1}}{2} - \frac{I_{0,1} I_{0000,5}}{8} - \frac{3 I_{0,1} I_{0000,7}}{4} + \frac{I_{0,1} I_{0000,21}}{32} + I_{00000,2}$$

At level 6 we use signature dependent identities expressing `RInv[{0,0,0,0}, 21]` in terms of `DualRInv[{0,0}, 2]` squared:

```
In[107]:=
      InvSimplify[rinv, 6] // Expand
```

Out[107]=

$$-\frac{1}{512} D_{00,2}^2 I_{0,1} - \frac{49 I_{0,1}^5}{384} + \frac{91}{96} I_{0,1}^3 I_{00,1} - \frac{3}{16} I_{0,1} I_{00,1}^2 - \frac{1}{48} I_{0,1}^3 I_{00,2} + \frac{5}{32} I_{0,1} I_{00,1} I_{00,2} - \frac{1}{128} I_{0,1} I_{00,2}^2 + \frac{5}{3} I_{0,1}^2 I_{000,1} + \frac{I_{00,1} I_{000,1}}{6} + \frac{I_{00,2} I_{000,1}}{6} - \frac{I_{00,1} I_{000,2}}{2} - \frac{I_{00,2} I_{000,2}}{8} - \frac{1}{96} I_{0,1}^2 I_{000,5} + \frac{11 I_{0,1} I_{0000,1}}{8} - \frac{I_{0,1} I_{0000,7}}{2} + I_{00000,2}$$

Another example, with derivatives and involving dual relations

```
In[108]:=
      diffrinv = RInv[metric][{0, 0, 1, 1}, 800]
```

Out[108]=

$$I_{0011,800}$$

```
In[109]:=
      InvSimplify[diffrinv, 2]
```

Out[109]=

$$-\frac{I_{0011,793}}{2}$$

```
In[110]:=
      InvSimplify[diffrinv, 3]
```

Out[110]=

$$-\frac{I_{0011,368}}{2}+\frac{I_{0011,369}}{2}$$

```
In[111]:=
      InvSimplify[diffrinv, 4]
```

Out[111]=

$$-\frac{I_{0011,368}}{2}+\frac{I_{0011,369}}{2}$$

```
In[112]:=
      InvSimplify[diffrinv, 5]
```

Out[112]=

$$-\frac{3}{32}I_{0,1}^2\,I_{11,1}+\frac{3\,I_{00,1}\,I_{11,1}}{8}-\frac{I_{00,2}\,I_{11,1}}{32}+\frac{3}{8}I_{0,1}^2\,I_{11,4}-\frac{3\,I_{00,1}\,I_{11,4}}{2}+\frac{I_{00,2}\,I_{11,4}}{8}-$$
$$\frac{3}{8}I_{0,1}^2\,I_{11,5}+\frac{3\,I_{00,1}\,I_{11,5}}{2}-\frac{I_{00,2}\,I_{11,5}}{8}+\frac{I_{0,1}\,I_{011,1}}{8}+\frac{I_{0,1}\,I_{011,2}}{2}+\frac{I_{0,1}\,I_{011,3}}{2}+$$
$$I_{0,1}\,I_{011,8}-\frac{I_{0,1}\,I_{011,9}}{2}-I_{0,1}\,I_{011,10}-\frac{I_{0,1}\,I_{011,11}}{2}+\frac{I_{0011,1}}{4}+I_{0011,2}+I_{0011,3}+$$
$$2\,I_{0011,10}-I_{0011,11}-2\,I_{0011,12}-I_{0011,13}-\frac{I_{0011,23}}{4}-I_{0011,24}-I_{0011,25}-2\,I_{0011,42}+$$
$$I_{0011,43}+2\,I_{0011,44}+I_{0011,50}+\frac{I_{0011,125}}{2}+\frac{I_{0011,362}}{2}-I_{0011,363}+\frac{I_{0011,367}}{2}-\frac{I_{0011,368}}{2}$$

```
In[113]:=
      InvSimplify[diffrinv, 6]
```

Out[113]=

$$-\frac{1}{16}D_{00,2}\,D_{11,1}-\frac{I_{0011,367}}{2}$$

Simplification of a dual invariant

```
In[114]:=
      dualrinv = DualRInv[metric][{1, 3}, 3]
```

Out[114]=

$$D_{13,3}$$

In[115]:=
```
InvSimplify[dualrinv, 2]
```
```
    Reading DualInvRules for step 2 and case {1, 3}
```
Out[115]=
$D_{13,3}$

In[116]:=
```
InvSimplify[dualrinv, 3]
```
```
    Reading DualInvRules for step 3 and case {1, 3}
```
Out[116]=
$$\frac{D_{13,1}}{2}$$

In[117]:=
```
InvSimplify[dualrinv, 4]
```
```
    Reading DualInvRules for step 4 and case {1, 3}
```
Out[117]=
0

By default, InvSimplify applies the first 6 steps. This behaviour can be changed with the following global variable

In[118]:=
```
$InvSimplifyLevel = 5
```
Out[118]=
5

In[119]:=
```
InvSimplify[rinv]
```
Out[119]=
$$-\frac{11\,I_{0,1}^5}{96}+\frac{79}{96}I_{0,1}^3\,I_{00,1}-\frac{1}{8}I_{0,1}\,I_{00,1}^2-\frac{1}{192}I_{0,1}^3\,I_{00,2}+\frac{5}{32}I_{0,1}\,I_{00,1}\,I_{00,2}-\frac{1}{64}I_{0,1}\,I_{00,2}^2+\frac{5}{3}I_{0,1}^2\,I_{000,1}+\frac{I_{00,1}\,I_{000,1}}{6}+\frac{I_{00,2}\,I_{000,1}}{6}+\frac{1}{4}I_{0,1}^2\,I_{000,2}-\frac{I_{00,1}\,I_{000,2}}{2}-\frac{I_{00,2}\,I_{000,2}}{8}-\frac{1}{32}I_{0,1}^2\,I_{000,5}+\frac{3\,I_{0,1}\,I_{0000,1}}{2}-\frac{I_{0,1}\,I_{0000,5}}{8}-\frac{3\,I_{0,1}\,I_{0000,7}}{4}+\frac{I_{0,1}\,I_{0000,21}}{32}+I_{00000,2}$$

In[120]:=
```
$InvSimplifyLevel = 2
```
Out[120]=
2

In[121]:=
```
InvSimplify[rinv]
```
Out[121]=
$-I_{00000,97}+I_{00000,99}$

We return to the default value

```
In[122]:=
      $InvSimplifyLevel = 6

Out[122]=
      6
```

Covariant derivatives commute when acting on scalars, which would in principle give an extra permutation symmetry. This symmetry is, however, not applied at step 1, but at step 4, along with the general rules for reordering of derivatives. For example, the following two invariants are obviously equal

```
In[123]:=
      InvToRiemann[RInv[metric][{4}, 2]]

         Reading InvRules for step 1 and case {4}

Out[123]=
```
Scalar$[R^{ab}{}_{ab}{}^{;c;d}{}_{;d;c}]$

```
In[124]:=
      InvToRiemann[RInv[metric][{4}, 3]]

Out[124]=
```
Scalar$[R^{ab}{}_{ab}{}^{;c;d}{}_{;c;d}]$

Nevertheless, they are considered different objects until step 4

```
In[125]:=
      InvSimplify[RInv[metric][{4}, 2] - RInv[metric][{4}, 3], 3]

Out[125]=
```
$I_{4,2} - I_{4,3}$

```
In[126]:=
      InvSimplify[RInv[metric][{4}, 2] - RInv[metric][{4}, 3], 4]

Out[126]=
      0
```

6.2. RiemannSimplify

This function combines most of the functionalities of `Invar'` and it is certainly the most important tool for the user. Essentially `RiemannSimplify` is equivalent to the process:

rexpr −> (**RiemannToInv**) −> rinv −> (**InvSimplify**) −> canon rinv −> (**InvToRiemann**) −> canon rexpr

This means that RiemannSimplify requires three additional arguments:

− a metric for `RiemannToInv`

− a simplification level for `InvSimplify`

− the curvature−relations switch to Ricci.

All three of them have default values, so that the function can be used with a single argument.

Example:

```
In[127]:=
      rexpr = RandomRiemannMonomial[{0, 2, 2}]
```

Out[127]=

$$\mathrm{Scalar}[R_{h}{}^{b}{}_{ef}\, R^{g}{}_{g}{}^{cd;e}{}_{;a}\, R_{db}{}^{fh;a}{}_{;c}]$$

```
In[128]:=
      RiemannSimplify[%]

Out[128]=
      0

In[129]:=
      ?RiemannSimplify

         RiemannSimplify[expr, level, cr, g] simplifies the Riemann scalars
           of metric g using relations up to the given level (see usage
           message for InvSimplify). With cr=True contracted Riemann tensors
           are replaced by Ricci tensors. If g is a list of metrics then
           the command is folded over the list. RiemannSimplify[expr] uses
           the defaults $InvSimplifyLevel, $CurvatureRelations and $Metrics,
           respectively. See usage messages for those global variables.

In[130]:=
      RiemannSimplify[RandomRiemannMonomial[{6}], 4, True, metric]

         Reading InvRules for step 1 and case {6}

         Reading InvRules for step 1 and case {1, 3}

         Reading InvRules for step 1 and case {2, 2}
```

Out[130]=

$$\begin{aligned}
&-4\,\mathrm{Scalar}[R^{ba} R_{de} R^{e}{}_{c} R_{a}{}^{c}{}_{b}{}^{d}] - 5\,\mathrm{Scalar}[R^{ba} R_{ef} R_{a}{}^{cde} R_{bcd}{}^{f}] + \\
&4\,\mathrm{Scalar}[R^{ba} R_{cf} R_{a}{}^{cde} R_{b}{}^{f}{}_{de}] + \mathrm{Scalar}[R^{ba} R^{c}{}_{a} R_{b}{}^{def} R_{cdef}] + \\
&4\,\mathrm{Scalar}[R^{ba} R_{ef} R_{a}{}^{c}{}_{b}{}^{d} R_{c}{}^{e}{}_{d}{}^{f}] + \frac{5}{2}\,\mathrm{Scalar}[R_{bc} R^{;c;b;a}{}_{;a}] + 6\,\mathrm{Scalar}[R^{a}{}_{d} R_{bc} R^{cd;b}{}_{;a}] - \\
&5\,\mathrm{Scalar}[R^{a}{}_{d} R_{bc} R^{cb;d}{}_{;a}] + 4\,\mathrm{Scalar}[R^{;c;b}{}_{;a} R_{bc}{}^{;a}] + 12\,\mathrm{Scalar}[R_{cd} R^{cb}{}_{;a} R_{b}{}^{d;a}] + \\
&\mathrm{Scalar}[R^{ba} R^{dc}{}_{;a} R_{cd;b}] + \mathrm{Scalar}[R^{a}{}_{c} R^{bc}{}_{;a} R_{;b}] + \frac{1}{4}\,\mathrm{Scalar}[R^{ba} R_{;a} R_{;b}] + \\
&6\,\mathrm{Scalar}[R_{ca} R_{d}{}^{a} R^{dc;b}{}_{;b}] + \frac{5}{4}\,\mathrm{Scalar}[R^{;a} R_{;a}{}^{;b}{}_{;b}] - \mathrm{Scalar}[R_{cd} R^{dc;a}{}_{;a}{}^{;b}{}_{;b}] + \\
&4\,\mathrm{Scalar}[R_{cd} R^{c}{}_{b;a} R^{ad;b}] - 2\,\mathrm{Scalar}[R^{a}{}_{d} R^{c}{}_{b;a} R_{c}{}^{d;b}] - 5\,\mathrm{Scalar}[R^{dc}{}_{;a;b} R_{cd}{}^{;a;b}] + \\
&8\,\mathrm{Scalar}[R^{d}{}_{a}{}^{;c}{}_{;b} R_{cd}{}^{;a;b}] + \mathrm{Scalar}[R_{;a;b} R^{;a;b}] + 2\,\mathrm{Scalar}[R^{;a;b} R_{ab}{}^{;c}{}_{;c}] + \\
&\frac{1}{2}\,\mathrm{Scalar}[R^{;a}{}_{;a}{}^{;b}{}_{;b}{}^{;c}{}_{;c}] + \mathrm{Scalar}[R_{de} R_{ac;b} R^{eadb;c}] - 9\,\mathrm{Scalar}[R_{de} R_{ab;c} R^{eadb;c}] - \\
&\mathrm{Scalar}[R_{bc}{}^{;d}{}_{;a} R^{a}{}_{d}{}^{;b;c}] - 8\,\mathrm{Scalar}[R^{abcd} R^{e}{}_{a;c} R_{be;d}] - \frac{1}{2}\,\mathrm{Scalar}[R^{abcd} R_{ac;b} R_{;d}] - \\
&3\,\mathrm{Scalar}[R_{e}{}^{a} R_{a}{}^{bcd} R^{e}{}_{c;b;d}] + \mathrm{Scalar}[R^{ba} R_{a}{}^{c}{}_{b}{}^{d} R_{;c;d}] - \mathrm{Scalar}[R^{ba;c}{}_{;c} R_{ab}{}^{;d}{}_{;d}] + \\
&6\,\mathrm{Scalar}[R^{ba;c} R_{ac;b}{}^{;d}{}_{;d}] - 6\,\mathrm{Scalar}[R^{ba;c} R_{ab;c}{}^{;d}{}_{;d}] - 4\,\mathrm{Scalar}[R^{ba} R_{a}{}^{c}{}_{b}{}^{d} R_{cd}{}^{;e}{}_{;e}] - \\
&\mathrm{Scalar}[R^{abcd} R_{ac;b;d}{}^{;e}{}_{;e}] - 4\,\mathrm{Scalar}[R^{abcd} R_{bd;e} R_{ac}{}^{;e}] - 2\,\mathrm{Scalar}[R_{ac;b;d;e} R^{abcd;e}] - \\
&2\,\mathrm{Scalar}[R_{a}{}^{e}{}_{c}{}^{f} R^{abcd} R_{be;d;f}] + 2\,\mathrm{Scalar}[R_{ab}{}^{ef} R^{abcd} R_{ce;d;f}] + 2\,\mathrm{Scalar}[R_{a}{}^{e}{}_{c}{}^{f} R^{abcd} R_{bd;e;f}]
\end{aligned}$$

6.3. Tests

This is a collection of 54 examples, all of which must return 0.

First we add several indices:

```
In[131]:=
      AddIndices[TangentM, {a1, a10, a11, a2, a20, a22, a3, a30, a4, a40, a5, a50, a6, a60, a7,
        a8, a9, b0, i1, i10, i11, i2, i20, i22, i3, i30, i4, i40, i5, i50, i6, i60, i7, i8, i9}]
```

Shortcuts:

```
In[132]:=
      inv = RInv[metric];
      dinv = DualRInv[metric];
```

Test of **RiemannToInv** and **InvToRiemann**:

```
In[134]:=
      test[1] = CD[-h]@CD[-a]@RiemannCD[a, b, c, d] RiemannCD[-b, -d, e, f]
          CD[-g]@CD[-c]@RiemannCD[-e, -f, g, h] + inv[{0, 2, 2}, 1157];
      test[2] = RiemannCD[-a, e, f, g] RiemannCD[a, b, c, d]
          RiemannCD[h, i, j, k] CD[-d][RiemannCD[-h, -i, -j, -k]]
          CD[-e][RiemannCD[-b, -c, -f, -g]] - inv[{0, 0, 0, 1, 1}, 27015];
      test[3] = RiemannCD[a, b, c, d] CD[-h]@CD[-g]@CD[-c]@CD[h]@
              CD[-f]@CD[-e]@CD[-d]@CD[g][RiemannCD[-a, e, -b, f]] + inv[{0, 8}, 20018];
      test[4] = epsilonmetric[-b, -d, -g, -h] CD[-c]@CD[-a]@CD[-f][RiemannCD[g, f, -e, h]]
          CD[e][RiemannCD[a, b, c, d]] - dinv[{1, 3}, 914];

In[138]:=
      RiemannToInv /@ Array[test, 4]

          Reading InvRules for step 1 and case {0, 2, 2}

          Reading InvRules for step 1 and case {0, 0, 0, 1, 1}

          Reading InvRules for step 1 and case {0, 8}

          Reading DualInvRules for step 1 and case {1, 3}

Out[138]=
      {0, 0, 0, 0}

In[139]:=
      InvToRiemann /@ Array[test, 4] // NoScalar // ToCanonical

Out[139]=
      {0, 0, 0, 0}
```

More tests of **RiemannToInv**, with algebraic invariants:

```
In[140]:=
      test[5] := inv[7, 306] -
         RiemannCD[a2, i1, -a2, i2] * RiemannCD[-i1, i3, -i3, i4] * RiemannCD[-i2, i5, i6, i7] *
          RiemannCD[-i4, i8, i9, i10] * RiemannCD[-i5, a1, -i8, -a1] *
          RiemannCD[-i6, a4, -i9, a3] * RiemannCD[-i7, -a4, -i10, -a3];
      test[6] := inv[1, 1] * dinv[2, 1] - (RiemannCD[i11, i22, -i11, -i22]) *
           (RiemannCD[i1, i2, -i1, i3] * RiemannCD[-i2, i4, i5, i6] *
             epsilonmetric[-i3, -i4, -i5, -i6]);
      test[7] := inv[3, 3] * dinv[5, 2] - (RiemannCD[b, i20, -b, i30] *
             RiemannCD[-i20, i40, i50, i60] * RiemannCD[-i30, -i40, -i50, -i60]) *
           (RiemannCD[a2, i1, -a2, i2] * RiemannCD[-i1, i3, -i3, i4] *
             RiemannCD[-i2, i5, -i5, i6] * RiemannCD[-i4, i7, i8, i9] *
             RiemannCD[-i7, i10, -i10, a1] * epsilonmetric[-i6, -i8, -i9, -a1]);
      test[8] := inv[6, 137] + 2 * dinv[5, 289] - RiemannCD[a2, i1, -a2, i2] *
          RiemannCD[-i1, i3, -i3, i4] * RiemannCD[-i2, i5, i6, i7] * RiemannCD[-i4, i8,
            -i6, i9] * RiemannCD[-i5, -i8, i10, a1] * RiemannCD[-i7, -i10, -i9, -a1] -
         2 * (RiemannCD[a2, i1, -a2, i2] * RiemannCD[-i1, i3, i4, i5] *
             RiemannCD[-i2, i6, -i4, i7] * RiemannCD[-i3, i8, -i7, i9] *
             RiemannCD[-i8, i10, -i10, a1] * epsilonmetric[-i5, -i6, -i9, -a1]);

In[144]:=
      RiemannToInv /@ Array[test, 4, 5] // AbsoluteTiming

Out[144]=
      {0.461524 Second, {0, 0, 0, 0}}
```

In this case we need to guide the computation due to the products of epsilon tensors:

```
In[145]:=
      test[9] := dinv[2, 3]^2 - (RiemannCD[a1, a2, a4, a3] * RiemannCD[-a1, -a4, a6, a5] *
             epsilonmetric[-a2, -a3, -a6, -a5]) * (RiemannCD[a10, a20, a40, a30] *
             RiemannCD[-a10, -a40, a60, a50] * epsilonmetric[-a20, -a30, -a60, -a50]);

In[146]:=
      test[9] // ExpandGdelta // ContractMetric // RiemannToInv // InvSimplify // Simplify

Out[146]=
      0
```

Tests of **RiemannSimplify**:

```
In[147]:=
     test[10] :=
       epsilonmetric[a, b, c, d] * RiemannCD[-a, -b, e, f] * RiemannCD[-c, -e, -f, g] *
         RiemannCD[-d, h, i, j] * RicciCD[-g, -i] * RicciCD[-h, -j] + 1 / 8 *
          (epsilonmetric[a, b, c, d] * RiemannCD[-a, -b, e, f] * RiemannCD[-c, -d, -e, -f]) *
          (RiemannCD[g, h, i, j] * RicciCD[-g, -i] * RicciCD[-h, -j]);
     test[11] := RiemannCD[a1, a2, -a1, a3] * RiemannCD[-a2, a4, -a3, a5] *
        RiemannCD[-a4, a6, a7, a8] * epsilonmetric[-a5, -a6, -a7, -a8];
     test[12] := RiemannCD[a1, a2, a3, -a1] * RiemannCD[-a3, a5, -a2, a4] *
        RiemannCD[-a4, a6, a8, a7] * epsilonmetric[-a5, -a7, -a8, -a6];
     test[13] := RiemannCD[a1, a2, -a1, a3] * RiemannCD[-a2, a4, a5, a6] *
        RiemannCD[-a3, -a4, a7, a8] * RiemannCD[-a5, b0, -b0, a9] *
        epsilonmetric[-a6, -a7, -a8, -a9] + 1 / 2 * RiemannCD[a11, a22, -a11, -a22] *
        RiemannCD[a1, a2, -a1, a3] * RiemannCD[-a2, a4, a5, a6] * RiemannCD[-a4, a7, -a7, a8] *
        epsilonmetric[-a3, -a5, -a6, -a8] - 1 / 16 * RiemannCD[a10, a20, -a10, -a20] *
        RiemannCD[a11, a22, -a11, -a22] * RiemannCD[a1, a2, a3, a4] *
        RiemannCD[-a1, -a2, a5, a6] * epsilonmetric[-a3, -a4, -a5, -a6];
     test[14] := -RicciCD[-a2, a3] * RicciCD[a5, a2] * RiemannCD[-a3, a6, a7, a8] *
        RiemannCD[-a5, -a7, -a6, b0] * RicciCD[-a8, -b0] + 5 / 24 * RicciScalarCD[] ^5 +
       49 / 24 * RicciScalarCD[] ^2 * RicciCD[a3, -a2] * RicciCD[a2, a5] * RicciCD[-a5, -a3] +
       RicciCD[a2, a3] * RiemannCD[-a2, a4, -a3, a5] *
        RiemannCD[-a4, a6, -a5, a7] * RicciCD[-a6, b0] * RicciCD[-a7, -b0] +
       1 / 24 * RicciScalarCD[] ^3 * RiemannCD[a1, a2, a3, a4] * RiemannCD[-a1, -a2, -a3, -a4] -
       5 / 4 * RicciScalarCD[] * RicciCD[a2, a3] * RicciCD[-a2, a5] *
        RicciCD[-a3, a7] * RicciCD[-a5, -a7] + 3 / 4 * RicciScalarCD[] ^2 *
        RicciCD[a2, a3] * RiemannCD[-a2, a4, -a3, a5] * RicciCD[-a4, -a5] -
       13 / 8 * RicciScalarCD[] ^3 * RicciCD[a2, a3] * RicciCD[-a2, -a3] -
       1 / 2 * RicciScalarCD[] * RicciCD[a2, -a3] * RiemannCD[-a2, a4, a5, a6] *
        RiemannCD[a7, a3, -a6, -a5] * RicciCD[-a4, -a7] -
       1 / 24 * RiemannCD[a3, a4, a1, a2] * RiemannCD[-a1, -a2, -a3, -a4] * RicciCD[a20, a30] *
        RicciCD[-a20, a50] * RicciCD[-a30, -a50] + 5 / 8 * RicciScalarCD[] *
        RicciCD[a2, a3] * RicciCD[-a2, -a3] * RicciCD[a20, a30] * RicciCD[-a20, -a30];

In[151]:=
     RiemannSimplify /@ Array[test, 5, 10] // AbsoluteTiming

Out[151]=
     {0.434133 Second, {0, 0, 0, 0, 0}}
```

Tests of the cyclic symmetries:

```
In[152]:=
     test[15] := inv[2, 3] - 1 / 2 * inv[2, 2] - inv[3, 6] + 1 / 2 * inv[3, 5];
     test[16] := -inv[4, 38] + 1 / 4 * inv[4, 23] - inv[4, 26] + inv[4, 36];
     test[17] := inv[6, 31] - inv[5, 203] - inv[5, 200] + inv[5, 201] - 1 / 4 * inv[6, 27];
     test[18] := -inv[6, 1575] + 3 / 4 * inv[6, 1219] +
        1 / 2 * inv[6, 1223] + inv[6, 1572] - 3 / 2 * inv[6, 1221];
     test[19] := inv[7, 15138] - 1 / 2 * inv[7, 12749] - inv[6, 1524] +
        inv[6, 1523] - 1 / 2 * inv[6, 1161];
     test[20] := -inv[7, 16207] - 1 / 4 * inv[7, 12848] + inv[7, 16206];
     test[21] :=
       -inv[7, 16467] - inv[7, 16383] - 4 * inv[7, 16269] - inv[7, 16280] - inv[7, 16306] -
        3 * inv[7, 16262] + 3 * inv[7, 16281] + 4 * inv[7, 16263] + inv[7, 16265] + inv[7, 16268] -
        3 * inv[7, 16318] - inv[7, 16325] + 2 * inv[7, 16292] - 2 * inv[7, 16341];
     test[22] := dinv[2, 1] + dinv[3, 1];
     test[23] := -dinv[3, 27] - 1 / 4 * dinv[3, 14] + 1 / 2 * dinv[3, 23] + dinv[4, 1] -
        dinv[4, 3] + 1 / 2 * dinv[4, 2] - 7 * dinv[4, 9] + dinv[4, 11] - 1 / 2 * dinv[4, 10];
     test[24] := dinv[4, 231] + dinv[4, 232] + dinv[5, 1] + dinv[5, 3] -
        1 / 2 * dinv[5, 2] + 10 * dinv[5, 8] + dinv[5, 9] - dinv[5, 10];
     test[25] := -dinv[5, 2560] + 2 * dinv[5, 2505] - dinv[5, 2507] - dinv[5, 2551];
```

```
In[163]:=
     $InvSimplifyLevel = 2;

In[164]:=
     InvSimplify /@ Array[test, 11, 15] // AbsoluteTiming

Out[164]=
     {0.081885 Second, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}

In[165]:=
     test[26] = inv[{1, 1, 4}, 15073] + inv[{1, 1, 4}, 15063] / 4 - inv[{1, 1, 4}, 15072];
     test[27] = dinv[{1, 3}, 914] - dinv[{1, 3}, 908] / 4;
     test[28] = inv[{0, 0, 0, 1, 1}, 27015] + 2 inv[{0, 0, 0, 1, 1}, 26365];
     test[29] = inv[{8}, 1116] + inv[{8}, 1092] - inv[{8}, 1115];

In[169]:=
     InvSimplify /@ Array[test, 4, 26]

        Reading InvRules for step 2 and case {1, 1, 4}

        Reading InvRules for step 2 and case {0, 0, 0, 1, 1}

Out[169]=
     {0, 0, 0, 0}
```

Tests of the Bianchi relations:

```
In[170]:=
     test[30] = inv[{1, 1, 4}, 15072] - inv[{1, 1, 4}, 1750] +
        inv[{1, 1, 4}, 1751] - 2 inv[{1, 1, 4}, 1753] - inv[{1, 1, 4}, 1754];
     test[31] = dinv[{1, 3}, 908] + 4 dinv[{1, 3}, 328] - 4 dinv[{1, 3}, 332];
     test[32] = inv[{0, 0, 0, 1, 1}, 26365] +
        4 inv[{0, 0, 0, 1, 1}, 26327] - 4 inv[{0, 0, 0, 1, 1}, 26335];
     test[33] = inv[{8}, 1092] + inv[{8}, 1090] - inv[{8}, 1091];

In[174]:=
     $InvSimplifyLevel = 3;

In[175]:=
     InvSimplify /@ Array[test, 4, 30]

        Reading InvRules for step 3 and case {1, 1, 4}

        Reading InvRules for step 3 and case {0, 0, 0, 1, 1}

Out[175]=
     {0, 0, 0, 0}
```

Tests of the commutation relations:

```
In[176]:=
      test[34] = inv[{0, 0, 4}, 70] - inv[{0, 0, 0, 2}, 50] - inv[{0, 0, 1, 1}, 100];
      test[35] = -inv[{0, 0, 4}, 74] + 2 inv[{0, 0, 4}, 18] - 2 inv[{0, 0, 4}, 19] +
         4 inv[{0, 0, 1, 1}, 71] - 4 inv[{0, 0, 1, 1}, 251] - 4 inv[{0, 0, 1, 1}, 264] +
         4 inv[{0, 0, 1, 1}, 267] - 2 inv[{0, 0, 0, 2}, 5] + 4 inv[{0, 0, 0, 2}, 6] -
         2 inv[{0, 0, 0, 2}, 8] - 2 inv[{0, 0, 0, 2}, 12] - 4 inv[{0, 0, 0, 2}, 31] +
         8 inv[{0, 0, 0, 2}, 32] - 4 inv[{0, 0, 0, 2}, 34] + 8 inv[{0, 0, 0, 2}, 35] +
         8 inv[{0, 0, 0, 2}, 37] - 4 inv[{0, 0, 0, 2}, 41] - 4 inv[{0, 0, 0, 2}, 42] +
         4 inv[{0, 0, 0, 0, 0}, 2] + 10 inv[{0, 0, 0, 0, 0}, 6] + 4 inv[{0, 0, 0, 0, 0}, 8] -
         4 inv[{0, 0, 0, 0, 0}, 9] + 8 inv[{0, 0, 0, 0, 0}, 11] + 6 inv[{0, 0, 0, 0, 0}, 12] +
         4 inv[{0, 0, 0, 0, 0}, 25] + 4 inv[{0, 0, 0, 0, 0}, 27] + 8 inv[{0, 0, 0, 0, 0}, 31] +
         8 inv[{0, 0, 0, 0, 0}, 33] + 8 inv[{0, 0, 0, 0, 0}, 35] - 8 inv[{0, 0, 0, 0, 0}, 36] -
         8 inv[{0, 0, 0, 0, 0}, 38] - 8 inv[{0, 0, 0, 0, 0}, 43] - 8 inv[{0, 0, 0, 0, 0}, 45];
      test[36] = -dinv[{1, 3}, 11] + dinv[{1, 3}, 10] + dinv[{0, 1, 1}, 2] - dinv[{0, 1, 1}, 4] -
         2 dinv[{0, 1, 1}, 6] + dinv[{0, 1, 1}, 88] - 2 dinv[{0, 1, 1}, 89];
      test[37] = dinv[{1, 3}, 5] - dinv[{0, 1, 1}, 1] - dinv[{0, 1, 1}, 18] / 2;

In[180]:=
      $InvSimplifyLevel = 4;

In[181]:=
      InvSimplify /@ Array[test, 4, 34]

         Reading DualInvRules for step 2 and case {0, 1, 1}

         Reading DualInvRules for step 3 and case {0, 1, 1}

         Reading DualInvRules for step 4 and case {0, 1, 1}

Out[181]=
      {0, 0, 0, 0}
```

Tests of the dimension dependent relations:

In[182]:=
```
test[38] := -inv[3, 3] + 1 / 4 * inv[1, 1] ^3 - 2 * inv[3, 1] +
   1 / 4 * inv[1, 1] * inv[2, 2] + 2 * inv[3, 2] - 2 * inv[1, 1] * inv[2, 1];
test[39] := -inv[4, 36] - 4 / 3 * inv[1, 1] * inv[3, 1] - 1 / 4 * inv[2, 1] * inv[2, 2] -
   3 * inv[4, 1] + 4 * inv[4, 7] + inv[4, 5] - 1 / 4 * inv[4, 21] + 3 / 4 * inv[1, 1] ^2 * inv[2, 1] -
   3 * inv[1, 1] * inv[3, 2] + 1 / 4 * inv[1, 1] * inv[3, 5] - 1 / 12 * inv[1, 1] ^4 -
   1 / 8 * inv[1, 1] ^2 * inv[2, 2] + 1 / 8 * inv[2, 2] ^2 - inv[5, 1] +
   1 / 24 * inv[1, 1] ^5 - 5 / 6 * inv[1, 1] ^2 * inv[3, 1] + 5 / 8 * inv[1, 1] * inv[2, 1] ^2 -
   5 / 4 * inv[1, 1] * inv[4, 1] - 5 / 12 * inv[1, 1] ^3 * inv[2, 1] + 5 / 6 * inv[2, 1] * inv[3, 1];
test[40] := -inv[6, 108] - 31 / 24 * inv[1, 1] ^3 * inv[3, 1] -
   11 / 16 * inv[1, 1] ^4 * inv[2, 1] + 1 / 12 * inv[1, 1] ^6 - 7 / 8 * inv[1, 1] ^2 * inv[4, 1] +
   7 / 16 * inv[1, 1] ^2 * inv[2, 1] ^2 + 1 / 8 * inv[1, 1] ^3 * inv[3, 2] - inv[1, 1] * inv[5, 2] +
   1 / 2 * inv[6, 47] + inv[6, 13] + inv[3, 1] ^2 - inv[3, 1] * inv[3, 2] +
   1 / 8 * inv[3, 1] * inv[3, 5] + 1 / 4 * inv[1, 1] * inv[2, 1] * inv[3, 2] +
   3 / 2 * inv[1, 1] * inv[2, 1] * inv[3, 1] - 1 / 8 * inv[1, 1] * inv[2, 2] * inv[3, 1];
test[41] := -inv[7, 7490] + 1 / 64 * inv[1, 1] * inv[2, 2] * inv[4, 21] + 1 / 2 * inv[7, 2190] -
   3 / 8 * inv[1, 1] * inv[2, 2] * inv[4, 1] + 1 / 12 * inv[2, 1] * inv[2, 2] * inv[3, 1] -
   23 / 48 * inv[1, 1] ^2 * inv[2, 2] * inv[3, 1] + 3 / 8 * inv[1, 1] * inv[2, 1] ^2 * inv[2, 2] -
   31 / 96 * inv[1, 1] ^3 * inv[2, 1] * inv[2, 2] - 1 / 2 * inv[1, 1] * inv[2, 1] * inv[4, 1] -
   11 / 12 * inv[1, 1] ^2 * inv[2, 1] * inv[3, 1] - 1 / 16 * inv[1, 1] * inv[2, 1] * inv[2, 2] ^2 -
   2 * inv[7, 14] + 1 / 192 * inv[1, 1] ^3 * inv[2, 2] ^2 + 5 / 12 * inv[1, 1] ^3 * inv[4, 1] -
   1 / 3 * inv[3, 1] * inv[4, 1] - 11 / 288 * inv[1, 1] ^7 - 25 / 48 * inv[1, 1] ^3 * inv[2, 1] ^2 +
   35 / 96 * inv[1, 1] ^5 * inv[2, 1] + 79 / 144 * inv[1, 1] ^4 * inv[3, 1] +
   1 / 6 * inv[2, 1] ^2 * inv[3, 1] - 4 / 9 * inv[1, 1] * inv[3, 1] ^2 +
   7 / 192 * inv[1, 1] ^5 * inv[2, 2] - 1 / 24 * inv[2, 2] ^2 * inv[3, 1] -
   2 * inv[7, 55] - 1 / 192 * inv[1, 1] ^3 * inv[4, 21] - 2 * inv[1, 1] * inv[6, 47] -
   3 / 4 * inv[1, 1] ^2 * inv[5, 33] + 1 / 2 * inv[3, 2] * inv[4, 1] -
   1 / 6 * inv[1, 1] ^4 * inv[3, 2] - 1 / 4 * inv[2, 1] ^2 * inv[3, 2] +
   1 / 4 * inv[1, 1] ^2 * inv[5, 2] - 1 / 3 * inv[3, 1] * inv[4, 5] +
   1 / 6 * inv[1, 1] ^3 * inv[4, 5] - 3 / 4 * inv[1, 1] ^2 * inv[5, 8] +
   1 / 24 * inv[3, 1] * inv[4, 21] - 3 / 4 * inv[2, 2] * inv[5, 2] -
   1 / 2 * inv[1, 1] * inv[2, 1] * inv[4, 5] + 5 / 6 * inv[1, 1] * inv[3, 1] * inv[3, 2] +
   1 / 2 * inv[1, 1] ^2 * inv[2, 1] * inv[3, 2] + 1 / 16 * inv[1, 1] ^2 * inv[2, 2] * inv[3, 2] +
   1 / 4 * inv[2, 2] * inv[5, 8] + 1 / 16 * inv[2, 2] ^2 * inv[3, 2] -
   2 * inv[1, 1] * inv[6, 13] + inv[2, 1] * inv[5, 2] + 1 / 4 * inv[2, 2] * inv[5, 33];
test[42] := -dinv[3, 23] + 1 / 2 * dinv[3, 13] + dinv[3, 2] -
   3 / 8 * inv[1, 1] * dinv[2, 2] - dinv[4, 2] - 1 / 2 * inv[1, 1] * dinv[3, 2];
test[43] := -dinv[5, 1770] + 5 / 32 * inv[1, 1] ^3 * dinv[2, 2] -
   9 / 8 * inv[1, 1] * inv[2, 1] * dinv[2, 2] + 3 / 32 * inv[1, 1] * inv[2, 2] * dinv[2, 2] -
   inv[3, 1] * dinv[2, 2] + 3 / 4 * inv[3, 2] * dinv[2, 2] -
   1 / 16 * inv[3, 5] * dinv[2, 2] - dinv[5, 1404] - 1 / 8 * inv[2, 2] * dinv[3, 13];
```

In[188]:=
```
$InvSimplifyLevel = 5;
```

In[189]:=
```
InvSimplify /@ Array[test, 6, 38] // AbsoluteTiming
```

Out[189]=
```
{0.081822 Second, {0, 0, 0, 0, 0, 0}}
```

```
In[190]:=
      test[44] = -inv[{1, 2, 3}, 5861] + inv[{1, 2, 3}, 7] / 8 +
         inv[{1, 2, 3}, 49] / 4 + inv[{1, 2, 3}, 380] / 2 - inv[{1, 2, 3}, 392] / 2 -
         inv[{1, 2, 3}, 474] / 2 + inv[{1, 2, 3}, 480] / 2 - inv[{1, 2, 3}, 625] / 2 +
         inv[{1, 2, 3}, 631] / 2 + inv[{1, 2, 3}, 1356] - 2 inv[{1, 2, 3}, 1362] +
         inv[{1, 2, 3}, 1380] - inv[{1, 2, 3}, 1416] + inv[{1, 2, 3}, 1422] -
         inv[{1, 2, 3}, 2229] / 2 + inv[{1, 2, 3}, 2626] / 2 - 2 inv[{1, 2, 3}, 4985] -
         inv[{1, 2, 3}, 5804] + inv[{1, 2, 3}, 5816] + inv[{1, 2, 3}, 5837];
      test[45] = dinv[{0, 0, 2}, 95] + dinv[{0, 0}, 2] inv[{2}, 1] / 4;
      test[46] =
        -inv[{0, 3, 3}, 3901] + inv[{0}, 1] inv[{3, 3}, 7] / 8 - inv[{0}, 1] inv[{3, 3}, 150] / 2 +
         inv[{0}, 1] inv[{3, 3}, 155] / 2 - inv[{0, 3, 3}, 14] / 4 - inv[{0, 3, 3}, 101] -
         inv[{0, 3, 3}, 111] - 2 inv[{0, 3, 3}, 585] + inv[{0, 3, 3}, 590] +
         2 inv[{0, 3, 3}, 602] + inv[{0, 3, 3}, 718] + inv[{0, 3, 3}, 1144] +
         inv[{0, 3, 3}, 3567] - 2 inv[{0, 3, 3}, 3572] + 2 inv[{0, 3, 3}, 3875];
      test[47] = dinv[{0, 1, 1}, 4] + dinv[{0, 1, 1}, 1] / 2 +
         dinv[{0, 1, 1}, 2] - dinv[{0, 1, 1}, 3];

In[194]:=
      InvSimplify /@ Array[test, 4, 44]

         Reading InvRules for step 2 and case {1, 2, 3}

         Reading InvRules for step 3 and case {1, 2, 3}

         Reading NEInvRules for step 4 and case {1, 2, 3}

         Reading InvRules for step 5, case {1, 2, 3} and dimension 4

         Reading DualInvRules for step 2 and case {0, 0, 2}

         Reading DualInvRules for step 3 and case {0, 0, 2}

         Reading DualInvRules for step 4 and case {0, 0, 2}

         Reading DualInvRules for step 5 and case {0, 0, 2}

         Reading InvRules for step 2 and case {0, 3, 3}

         Reading InvRules for step 3 and case {0, 3, 3}

         Reading NEInvRules for step 4 and case {0, 3, 3}

         Reading InvRules for step 5, case {0, 3, 3} and dimension 4

         Reading DualInvRules for step 5 and case {0, 1, 1}

Out[194]=
      {0, 0, 0, 0}
```

Test of the signature dependent relations:

```
In[195]:=
      test[48] := -inv[5, 123] - 5 * dinv[2, 2] * dinv[3, 2] / 24 -
         5 * dinv[2, 2] * dinv[3, 13] / 48 - 2 * inv[1, 1] ^5 / 3 + 25 * inv[1, 1] ^3 * inv[2, 1] / 6 +
         15 * inv[1, 1] * inv[2, 1] ^2 / 2 - 5 * inv[1, 1] * inv[2, 1] * inv[2, 2] / 2 +
         15 * inv[1, 1] ^2 * inv[3, 1] / 2 + 10 * inv[2, 1] * inv[3, 1] / 3 -
         5 * inv[2, 2] * inv[3, 1] / 6 - 5 * inv[1, 1] ^2 * inv[3, 2] / 2 - 10 * inv[2, 1] * inv[3, 2] +
         5 * inv[2, 2] * inv[3, 2] / 2 + 5 * inv[1, 1] ^2 * inv[3, 5] / 12 - 5 * inv[2, 1] * inv[3, 5] / 6 +
         5 * inv[2, 2] * inv[3, 5] / 12 + 5 * inv[1, 1] * inv[4, 1] + 20 * inv[5, 8] + 20 * inv[5, 33];

In[196]:=
      $InvSimplifyLevel = 6;
```

```
In[197]:=
      test[48] // InvSimplify

Out[197]=
      0

In[198]:=
      test[49] = -inv[{1, 1, 1, 1}, 53] - dinv[{1, 1}, 1]^2 / 4 -
         inv[{1, 1}, 1]^2 / 64 + 3 inv[{1, 1}, 1] inv[{1, 1}, 4] / 8 -
         5 inv[{1, 1}, 4]^2 / 4 - 3 inv[{1, 1}, 1] inv[{1, 1}, 5] / 8 +
         5 inv[{1, 1}, 4] inv[{1, 1}, 5] / 2 - 5 inv[{1, 1}, 5]^2 / 4 -
         inv[{1, 1, 1, 1}, 3] / 4 + inv[{1, 1, 1, 1}, 5] / 4 + inv[{1, 1, 1, 1}, 12] -
         inv[{1, 1, 1, 1}, 14] - inv[{1, 1, 1, 1}, 15] - inv[{1, 1, 1, 1}, 16] +
         inv[{1, 1, 1, 1}, 20] - 2 inv[{1, 1, 1, 1}, 22] - inv[{1, 1, 1, 1}, 23] -
         3 inv[{1, 1, 1, 1}, 38] + 2 inv[{1, 1, 1, 1}, 39] + 2 inv[{1, 1, 1, 1}, 40] +
         6 inv[{1, 1, 1, 1}, 41] - 4 inv[{1, 1, 1, 1}, 42] - inv[{1, 1, 1, 1}, 43] -
         inv[{1, 1, 1, 1}, 44] + inv[{1, 1, 1, 1}, 45] / 2 + 3 inv[{1, 1, 1, 1}, 46] / 2;
      test[50] = inv[{0, 0, 1, 1}, 1104] - dinv[{0, 0}, 2] dinv[{1, 1}, 25] / 4 -
         inv[{0, 0, 1, 1}, 1098] / 2;
      test[51] = -inv[{0, 0, 1, 3}, 25986] + dinv[{0, 0}, 2] dinv[{1, 3}, 294] / 8 +
         dinv[{0, 0}, 2] dinv[{0, 1, 1}, 1] / 8 + dinv[{0, 0}, 2] dinv[{0, 1, 1}, 2] / 4 +
         3 dinv[{0, 0}, 2] dinv[{0, 1, 1}, 18] / 32 + dinv[{0, 0}, 2] dinv[{0, 1, 1}, 63] / 8 -
         dinv[{0, 0}, 2] dinv[{0, 1, 1}, 86] / 2 + 5 dinv[{0, 0}, 2] dinv[{0, 1, 1}, 87] / 8 +
         dinv[{0, 0}, 2] dinv[{0, 1, 1}, 97] / 16 + dinv[{0, 0}, 2] dinv[{0, 1, 1}, 117] / 32 +
         3 dinv[{0, 0}, 2] dinv[{0, 1, 1}, 271] / 16 -
         dinv[{0, 0}, 2] dinv[{1, 1}, 1] inv[{0}, 1] / 16 + inv[{0, 0, 1, 3}, 25974] / 2;
      test[52] = -inv[{0, 0, 2, 2}, 5470] - dinv[{0, 0}, 2] dinv[{2, 2}, 19] / 8 -
         dinv[{0, 0}, 2] dinv[{0, 0, 2}, 2] / 4 + dinv[{0, 0}, 2] dinv[{0, 0, 2}, 5] / 16 -
         dinv[{0, 0}, 2] dinv[{0, 0, 2}, 14] / 16 + dinv[{0, 0}, 2] dinv[{0, 0, 2}, 32] / 4 -
         dinv[{0, 0}, 2] dinv[{0, 2}, 15] inv[{0}, 1] / 16 +
         dinv[{0, 0}, 2] dinv[{0, 0, 0}, 2] inv[{0}, 1] / 16 -
         dinv[{0, 0}, 2]^2 inv[{0}, 1]^2 / 128 + dinv[{0, 0}, 2]^2 inv[{0, 0}, 1] / 32 -
         inv[{0, 0, 2, 2}, 5461] + inv[{0, 0, 2, 2}, 5466];
```

```
In[202]:=
      InvSimplify /@ Array[test, 4, 49]

         Reading DualInvRules for step 2 and case {1, 1}

         Reading DualInvRules for step 3 and case {1, 1}

         Reading DualInvRules for step 4 and case {1, 1}

         Reading DualInvRules for step 5 and case {1, 1}

         Reading InvRules for step 2 and case {1, 1, 1, 1}

         Reading InvRules for step 3 and case {1, 1, 1, 1}

         Reading InvRules for step 4 and case {1, 1, 1, 1}

         Reading InvRules for step 5, case {1, 1, 1, 1} and dimension 4

         Reading InvRules for step 6, case {1, 1, 1, 1} and dimension 4

         Reading DualInvRules for step 5 and case {1, 3}

         Reading InvRules for step 2 and case {0, 0, 1, 3}

         Reading InvRules for step 3 and case {0, 0, 1, 3}

         Reading InvRules for step 4 and case {0, 0, 1, 3}

         Reading InvRules for step 5, case {0, 0, 1, 3} and dimension 4

         Reading InvRules for step 6, case {0, 0, 1, 3} and dimension 4

         Reading DualInvRules for step 2 and case {2, 2}

         Reading DualInvRules for step 3 and case {2, 2}

         Reading DualInvRules for step 4 and case {2, 2}

         Reading DualInvRules for step 5 and case {2, 2}

         Reading DualInvRules for step 2 and case {0, 2}

         Reading DualInvRules for step 3 and case {0, 2}

         Reading DualInvRules for step 4 and case {0, 2}

         Reading DualInvRules for step 5 and case {0, 2}

         Reading InvRules for step 2 and case {0, 0, 2, 2}

         Reading InvRules for step 3 and case {0, 0, 2, 2}

         Reading InvRules for step 4 and case {0, 0, 2, 2}

Out[202]=
      {0, 0, 0, 0}
```

Test of **WeylToRiemann**:

```
In[203]:=
      test[53] := WeylCD[a, b, c, d] - RiemannCD[a, b, c, d] -
         1 / 2 * metric[a, d] * RicciCD[b, c] - 1 / 2 * metric[b, c] * RicciCD[a, d] +
         1 / 2 * metric[a, c] * RicciCD[b, d] + 1 / 2 * metric[b, d] * RicciCD[a, c] -
         1 / 6 * RicciScalarCD[] * (metric[a, c] * metric[b, d] - metric[a, d] * metric[b, c]);
```

```
In[204]:=
      test[53] // WeylToRiemann // Simplify // AbsoluteTiming

Out[204]=
      {0.027630 Second, 0}
```

Test of **ContractCurvature**:

```
In[205]:=
      test[54] := RiemannCD[c, d, -c, -d] *
          RiemannCD[a, f, b, -f] * RiemannCD[-a, -b, h, l] * metric[-h, -l] -
        RicciScalarCD[] * RicciCD[a, b] * RiemannCD[-a, -b, h, l] * metric[-h, -l];

In[206]:=
      test[54] // ContractCurvature // ContractMetric // AbsoluteTiming

Out[206]=
      {0.004365 Second, 0}
```

Tidy up:

```
In[207]:=
      RemoveIndices[TangentM,
        {a1, a10, a11, a2, a20, a22, a3, a30, a4, a40, a5, a50, a6, a60, a7, a8, a9, b0,
         i1, i10, i11, i2, i20, i22, i3, i30, i4, i40, i5, i50, i6, i60, i7, i8, i9}];
      Remove[
       test]
```

6.4. Zero powers of Riemann

Surprisingly, there are monomials of the Riemann tensor which are zero due to the cyclic symmetry only. This first happens for a single monomial of degree 6:

It does not happen for degrees 1 to 5:

```
In[209]:=
      Cases[RInvRules[2, 1], HoldPattern[_ → 0]]

Out[209]=
      {}

In[210]:=
      Cases[RInvRules[2, 2], HoldPattern[_ → 0]]

Out[210]=
      {}

In[211]:=
      Cases[RInvRules[2, 3], HoldPattern[_ → 0]]

Out[211]=
      {}

In[212]:=
      Cases[RInvRules[2, 4], HoldPattern[_ → 0]]

Out[212]=
      {}
```

```
In[213]:=
      Cases[RInvRules[2, 5], HoldPattern[_ → 0]]

Out[213]=
      {}
```

There is a single case in degree 6:

```
In[214]:=
      Cases[RInvRules[2, 6], HoldPattern[_ → 0]]

Out[214]=
```

$\{I_{000000,1461} \to 0\}$

And 19 more cases for degree 7:

```
In[215]:=
      Cases[RInvRules[2, 7], HoldPattern[_ → 0]]

Out[215]=
```

$\{I_{0000000,663} \to 0,\ I_{0000000,1600} \to 0,\ I_{0000000,1601} \to 0,\ I_{0000000,3439} \to 0,\ I_{0000000,3440} \to 0,\ I_{0000000,3460} \to 0,\ I_{0000000,4987} \to 0,\ I_{0000000,4988} \to 0,\ I_{0000000,5822} \to 0,\ I_{0000000,6830} \to 0,\ I_{0000000,6831} \to 0,\ I_{0000000,7545} \to 0,\ I_{0000000,7546} \to 0,\ I_{0000000,10851} \to 0,\ I_{0000000,10852} \to 0,\ I_{0000000,11644} \to 0,\ I_{0000000,13382} \to 0,\ I_{0000000,13843} \to 0,\ I_{0000000,14054} \to 0\}$

There is no obvious reason why this should be zero, and for all dimensions:

```
In[216]:=
      RInv[metric][6, 1461] // InvToRiemann // NoScalar

Out[216]=
```

$R_{ac}{}^{ef}\, R^{abcd}\, R_{b}{}^{ghi}\, R_{d}{}^{j}{}_{h}{}^{k}\, R_{eij}{}^{l}\, R_{fkgl}$

but we can check it in the pure Ricci case:

```
In[217]:=
      expr =
       % /. RiemannCD[a_, b_, c_, d_] → RicciCD[a, c] delta[b, d] + delta[a, c] RicciCD[b, d] -
          RicciCD[a, d] delta[b, c] - delta[a, d] RicciCD[b, c]

Out[217]=
```

$(g^{bd} R^{ac} - g^{bc} R^{ad} - g^{ad} R^{bc} + g^{ac} R^{bd})\ (\delta_{c}{}^{f} R_{a}{}^{e} - \delta_{c}{}^{e} R_{a}{}^{f} - \delta_{a}{}^{f} R_{c}{}^{e} + \delta_{a}{}^{e} R_{c}{}^{f})$
$(g^{gi} R_{b}{}^{h} - g^{gh} R_{b}{}^{i} - \delta_{b}{}^{i} R^{gh} + \delta_{b}{}^{h} R^{gi})\ (\delta_{i}{}^{l} R_{ej} - g_{ij} R_{e}{}^{l} - \delta_{e}{}^{l} R_{ij} + g_{ej} R_{i}{}^{l})$
$(g^{jk} R_{dh} - \delta^{j}{}_{h} R_{d}{}^{k} - \delta_{d}{}^{k} R^{j}{}_{h} + g_{dh} R^{jk})\ (g_{kl} R_{fg} - g_{kg} R_{fl} - g_{fl} R_{kg} + g_{fg} R_{kl})$

```
In[218]:=
      expr[[1]] expr[[2]] // ContractMetric // ToCanonical

Out[218]=
```

$-2 R^{ad} R^{bc} + 2 R^{ac} R^{bd} + g^{ce} R^{ba} R^{d}{}_{a} - g^{be} R^{ca} R^{d}{}_{a} - g^{cd} R^{ba} R^{e}{}_{a} + g^{bd} R^{ca} R^{e}{}_{a}$

```
In[219]:=
      % expr[[3]] // ContractMetric // ToCanonical
```

Out[219]=

$$\begin{aligned}
& g^{eg} R^{ba} R^{cf} R^{d}{}_{a} - g^{ef} R^{ba} R^{cg} R^{d}{}_{a} - 2 R^{ac} R^{bf} R^{de} + 2 R^{ab} R^{cf} R^{de} + 2 R^{ac} R^{be} R^{df} - \\
& g^{eg} R^{ba} R^{c}{}_{a} R^{df} - 2 R^{ab} R^{ce} R^{df} + g^{ef} R^{ba} R^{c}{}_{a} R^{dg} + g^{dg} R^{ba} R^{c}{}_{a} R^{ef} - g^{cg} R^{ba} R^{d}{}_{a} R^{ef} - \\
& g^{df} R^{ba} R^{c}{}_{a} R^{eg} + g^{cf} R^{ba} R^{d}{}_{a} R^{eg} + 2 g^{eg} R^{bd} R^{ca} R^{f}{}_{a} - 2 g^{eg} R^{bc} R^{da} R^{f}{}_{a} + g^{bd} R^{ca} R^{eg} R^{f}{}_{a} - \\
& g^{bc} R^{da} R^{eg} R^{f}{}_{a} - 2 g^{ef} R^{bd} R^{ca} R^{g}{}_{a} + 2 g^{ef} R^{bc} R^{da} R^{g}{}_{a} - g^{bd} R^{ca} R^{ef} R^{g}{}_{a} + g^{bc} R^{da} R^{ef} R^{g}{}_{a} + \\
& g^{ce} g^{fh} R_{ab} R^{da} R^{gb} - g^{cd} g^{fh} R_{ab} R^{ea} R^{gb} - g^{ce} g^{fg} R_{ab} R^{da} R^{hb} + g^{cd} g^{fg} R_{ab} R^{ea} R^{hb}
\end{aligned}$$

```
In[220]:=
      % expr[[4]] // ContractMetric // ToCanonical
```

Out[220]=

$$\begin{aligned}
& 3 R^{bg} R^{ca} R^{d}{}_{f} R^{e}{}_{a} - 3 R^{b}{}_{f} R^{ca} R^{dg} R^{e}{}_{a} + 2 R^{ba} R^{c}{}_{a} R^{dg} R^{e}{}_{f} - 2 R^{ba} R^{c}{}_{a} R^{d}{}_{f} R^{eg} + 2 R^{bg} R^{ce} R^{da} R_{fa} - \\
& 3 R^{bg} R^{ca} R^{de} R_{fa} + 3 R^{ba} R^{cg} R^{de} R_{fa} - 3 R^{ba} R^{ce} R^{dg} R_{fa} + 4 R^{bc} R^{dg} R^{ea} R_{fa} - 2 R^{bc} R^{da} R^{eg} R_{fa} - \\
& g^{eh} R_{ab} R^{c}{}_{g} R^{da} R^{fb} + \delta^{e}{}_{g} R_{ab} R^{ch} R^{da} R^{fb} + g^{ch} R_{ab} R^{da} R^{e}{}_{g} R^{fb} - \delta^{c}{}_{g} R_{ab} R^{da} R^{eh} R^{fb} - \\
& 2 g^{eh} R^{ca} R^{db} R^{f}{}_{b} R_{ga} - g^{fh} R^{ca} R^{d}{}_{a} R^{eb} R_{gb} + g^{ch} R^{da} R^{eb} R^{f}{}_{a} R_{gb} + g^{eh} R^{ca} R^{d}{}_{a} R^{fb} R_{gb} - \\
& 3 g^{ef} R_{ab} R^{ch} R^{da} R_{g}{}^{b} - g^{eh} R_{ab} R^{ca} R^{df} R_{g}{}^{b} + g^{ef} R_{ab} R^{ca} R^{dh} R_{g}{}^{b} + g^{dh} R_{ab} R^{ca} R^{ef} R_{g}{}^{b} - \\
& g^{ch} R_{ab} R^{da} R^{ef} R_{g}{}^{b} - g^{df} R_{ab} R^{ca} R^{eh} R_{g}{}^{b} + 2 g^{eh} R_{ab} R^{cd} R^{fa} R_{g}{}^{b} + 2 g^{cd} R_{ab} R^{eh} R^{fa} R_{g}{}^{b} - \\
& 2 R^{b}{}_{f} R^{ce} R^{da} R^{g}{}_{a} + 3 R^{b}{}_{f} R^{ca} R^{de} R^{g}{}_{a} - 3 R^{ba} R^{c}{}_{f} R^{de} R^{g}{}_{a} + 3 R^{ba} R^{ce} R^{d}{}_{f} R^{g}{}_{a} - \\
& 4 R^{bc} R^{d}{}_{f} R^{ea} R^{g}{}_{a} + 2 R^{bc} R^{da} R^{e}{}_{f} R^{g}{}_{a} - g^{ci} g^{fg} R_{a}{}^{d} R_{bd} R^{ea} R_{h}{}^{b} + g^{ce} g^{fi} R_{a}{}^{d} R_{bd} R^{ga} R_{h}{}^{b} + \\
& 2 \delta^{e}{}_{g} R^{ca} R^{db} R^{f}{}_{b} R^{h}{}_{a} - 3 g^{ef} R^{ca} R^{db} R_{gb} R^{h}{}_{a} + g^{df} R^{ca} R^{eb} R_{gb} R^{h}{}_{a} + g^{cd} R^{ea} R^{fb} R_{gb} R^{h}{}_{a} + \\
& \delta^{f}{}_{g} R^{ca} R^{d}{}_{a} R^{eb} R^{h}{}_{b} - \delta^{c}{}_{g} R^{da} R^{eb} R^{f}{}_{a} R^{h}{}_{b} - \delta^{e}{}_{g} R^{ca} R^{d}{}_{a} R^{fb} R^{h}{}_{b} + 3 g^{ef} R^{ca} R^{db} R_{ga} R^{h}{}_{b} - \\
& g^{df} R^{ca} R^{eb} R_{ga} R^{h}{}_{b} - g^{cd} R^{ea} R^{fb} R_{ga} R^{h}{}_{b} + 3 g^{ef} R_{ab} R^{c}{}_{g} R^{da} R^{hb} + \delta^{e}{}_{g} R_{ab} R^{ca} R^{df} R^{hb} - \\
& g^{ef} R_{ab} R^{ca} R^{d}{}_{g} R^{hb} - \delta^{d}{}_{g} R_{ab} R^{ca} R^{ef} R^{hb} + \delta^{c}{}_{g} R_{ab} R^{da} R^{ef} R^{hb} + g^{df} R_{ab} R^{ca} R^{e}{}_{g} R^{hb} - \\
& 2 \delta^{e}{}_{g} R_{ab} R^{cd} R^{fa} R^{hb} - 2 g^{cd} R_{ab} R^{e}{}_{g} R^{fa} R^{hb} + \delta^{c}{}_{h} g^{fg} R_{a}{}^{d} R_{bd} R^{ea} R^{ib} - \delta^{f}{}_{h} g^{ce} R_{a}{}^{d} R_{bd} R^{ga} R^{ib}
\end{aligned}$$

```
In[221]:=
      % expr[[5]] // ContractMetric // ToCanonical
```

Out[221]=

$$\begin{aligned}
& -g^{eh} g^{fi} R_{a}{}^{c} R^{ab} R_{b}{}^{d} R_{c}{}^{g} R_{dg} - 3 g^{fh} R_{a}{}^{c} R^{ab} R_{b}{}^{d} R_{cd} R^{eg} - 6 R_{a}{}^{c} R^{ab} R_{bc} R^{df} R^{eg} - \\
& 6 R_{bc} R^{bc} R^{da} R^{eg} R^{f}{}_{a} + 11 R_{a}{}^{c} R_{bc} R^{da} R^{eg} R^{fb} - 2 R_{ab} R^{c}{}_{c} R^{da} R^{eg} R^{fb} - 2 g^{eg} R_{a}{}^{c} R^{ab} R_{b}{}^{d} R_{cd} R^{fh} - \\
& 3 g^{fh} R_{b}{}^{d} R^{bc} R_{cd} R^{ea} R^{g}{}_{a} - 3 R_{bc} R^{bc} R^{df} R^{ea} R^{g}{}_{a} - 3 R^{c}{}_{c} R^{da} R^{eb} R^{f}{}_{a} R^{g}{}_{b} - \\
& 2 R_{ac} R^{da} R^{eb} R^{fc} R^{g}{}_{b} + 7 g^{fh} R_{a}{}^{c} R_{b}{}^{d} R_{cd} R^{ea} R^{gb} - g^{fh} R_{ab} R_{cd} R^{cd} R^{ea} R^{gb} + \\
& 7 R_{a}{}^{c} R_{bc} R^{df} R^{ea} R^{gb} + 2 R_{bc} R^{da} R^{eb} R^{f}{}_{a} R^{gc} - g^{eg} R_{b}{}^{d} R^{bc} R_{cd} R^{fa} R^{h}{}_{a} + 3 g^{eg} R_{a}{}^{c} R_{b}{}^{d} R_{cd} R^{fa} R^{hb}
\end{aligned}$$

```
In[222]:=
      % expr[[6]] // ContractMetric // ToCanonical
```

Out[222]=

```
      0
```

■ 7. Working with order 12 invariants

Fully expanded relations for the fourth step of order 12 invariants take up too much memory to be loaded at the same time in a typical PC. We provide two versions of the database for the hardest cases of this step and order:

− Non expanded: these rules have not been fully ‘simplified’ taking into account all the other relations. As a result, they are both somewhat smaller and slower to use. The corresponding files in the database end in the string ‘NE’.

− Expanded: these rules have been fully expanded. They are bigger but faster.

We can control which version to use with the global variable `$ExpandedCommuteOrder12Q`:

The default value is False, meaning that Invar will load the non expanded version.

```
In[223]:=
      $ExpandedCommuteOrder12Q

Out[223]=
      False
```

For example, let us take this invariant:

```
In[224]:=
      inv = RInv[metric][{0, 3, 3}, 100]

Out[224]=
```
$I_{033,100}$

which is still independent after steps 2 and 3:

```
In[225]:=
      InvSimplify[inv, 3]

Out[225]=
```
$I_{033,100}$

It can be expanded using the NE−rules for step 4:

```
In[226]:=
      InvSimplify[%, 4]

Out[226]=
```
$I_{033,28} + I_{0013,60} + 2\, I_{0013,480}$

but it is still not fully expanded, as this expression changes again under `InvSimplify`:

```
In[227]:=
      InvSimplify[%, 4]

Out[227]=
```
$I_{033,28} + I_{0013,22} + 2\, I_{0013,386}$

The rules take less than 20 Mbytes:

```
In[228]:=
      ByteCount[RInvRules[4, {0, 3, 3}]] / 1000. / 1024

Out[228]=
      19.7127
```

Now remove the rules

```
In[229]:=
      RemoveRInvRules[4, {0, 3, 3}]
```

and simplify again the invariant, now using the expanded rules. We get the fully expanded result

```
In[230]:=
      $ExpandedCommuteOrder12Q = True;
      InvSimplify[inv, 4]

          Reading InvRules for step 4 and case {0, 3, 3}
```

Out[231]=

$I_{033,28} + I_{0013,22} + 2\, I_{0013,386}$

but now the rules take more than 80 Mbytes:

```
In[232]:=
      ByteCount[RInvRules[4, {0, 3, 3}]] / 1000. / 1024

Out[232]=
      84.0526
```

Tidy up:

```
In[233]:=
      Clear[inv]
```

■ 8. The (algebraic) Narlikar and Karmarkar basis

8.1. Definitions

This section deals only with algebraic invariants. It is well known that there are 14 independent scalars of the Riemann tensor. Narlikar and Karmarkar gave 14 such scalars in 1947. We express them in terms of our basis of invariants.

Definitions taken from [Alex Harvey, Class. Quantum Grav. 7 (1990) 715−716]. First define six intermediate tensors:

```
In[234]:=
      IndexSet[Atensor[h_, i_, j_, k_], WeylCD[h, i, -a, -b] WeylCD[a, b, j, k]];
      IndexSet[Btensor[h_, i_, j_, k_], WeylCD[h, i, -a, -b] Atensor[a, b, j, k]];
      IndexSet[Dtensor[h_, i_, j_, k_],
        Btensor[h, i, j, k] - 1 / 12 J2 (metric[h, j] metric[i, k] - metric[h, k] metric[i, j]) -
         1 / 4 J1 WeylCD[h, i, j, k]];
      IndexSet[Etensor[h_, i_, j_, k_], WeylCD[h, i, -a, -b] Dtensor[a, b, j, k]];
      IndexSet[Ftensor[h_, i_, j_, k_], WeylCD[h, i, -a, -b] Etensor[a, b, j, k]];
      IndexSet[Qtensor[a_, b_], RicciCD[-c, a] RicciCD[c, b]];
```

These are the 14 scalars:

```
In[240]:=
      I1 := RicciScalarCD[];
      IndexSet[I2, RicciCD[-a, b] RicciCD[-b, a]];
      IndexSet[I3, RicciCD[-a, b] RicciCD[-b, c] RicciCD[-c, a]];
      IndexSet[I4, RicciCD[-a, b] RicciCD[-b, c] RicciCD[-c, d] RicciCD[-d, a]];
      IndexSet[J1, Atensor[-i, -j, i, j]];
      IndexSet[J2, Btensor[-i, -j, i, j]];
      IndexSet[J3, Etensor[-i, -j, i, j]];
      IndexSet[J4, Ftensor[-i, -j, i, j]];
      IndexSet[K1, WeylCD[-h, -i, -j, -k] RicciCD[h, j] RicciCD[i, k]];
      IndexSet[K2, Atensor[-h, -i, -j, -k] RicciCD[h, j] RicciCD[i, k]];
      IndexSet[K3, Dtensor[-h, -i, -j, -k] RicciCD[h, j] RicciCD[i, k]];
      IndexSet[K4, WeylCD[-h, -i, -j, -k] Qtensor[h, j] Qtensor[i, k]];
      IndexSet[K5, Atensor[-h, -i, -j, -k] Qtensor[h, j] Qtensor[i, k]];
      IndexSet[K6, Dtensor[-h, -i, -j, -k] Qtensor[h, j] Qtensor[i, k]];
```

Pure Ricci:

```
In[254]:=
      {I1, I2, I3, I4}
```

Out[254]=

$$\{R,\ R_{a}{}^{b} R_{b}{}^{a},\ R_{a}{}^{b} R_{b}{}^{c} R_{c}{}^{a},\ R_{a}{}^{b} R_{b}{}^{c} R_{c}{}^{d} R_{d}{}^{a}\}$$

Pure Weyl:

```
In[255]:=
      {J1, J2, J3, J4} // ContractMetric
```

Out[255]=

$$\Big\{W^{abcd} W_{cdab},\ W^{acef} W^{bd}{}_{ac} W_{efbd},\ -\frac{1}{4} W^{aceg} W^{bdfh} W_{egac} W_{fhbd} + W^{adgh} W^{be}{}_{ad} W^{cf}{}_{be} W_{ghcf},$$
$$-\frac{1}{4} W^{adhj} W^{begi} W^{cf}{}_{ad} W_{gibe} W_{hjcf} - \frac{1}{12} W^{aeij} W^{bf}{}_{dh} W^{cg}{}_{ae} W^{dh}{}_{bf} W_{ijcg} +$$
$$\frac{1}{12} W^{aeij} W^{cg}{}_{ae} W^{dh}{}_{bf} W^{fb}{}_{dh} W_{ijcg} + W^{aeij} W^{bf}{}_{ae} W^{cg}{}_{bf} W^{dh}{}_{cg} W_{ijdh}\Big\}$$

Mixed invariants:

```
In[256]:=
      {K1, K2, K3, K4, K5, K6} // ContractMetric
```

Out[256]=

$$\Big\{R^{ac} R^{bd} W_{abcd},\ R^{ce} R^{df} W^{ab}{}_{ef} W_{cdab},\ -\frac{1}{4} R^{cg} R^{eh} W^{abdf} W_{cegh} W_{dfab} + R^{eg} R^{fh} W^{ac}{}_{gh} W^{bd}{}_{ac} W_{efbd} +$$
$$\frac{1}{12} R^{e}{}_{g} R^{g}{}_{e} W^{acfh} W^{bd}{}_{ac} W_{fhbd} - \frac{1}{12} R^{e}{}_{e} R^{g}{}_{g} W^{acfh} W^{bd}{}_{ac} W_{fhbd},\ R_{a}{}^{c} R^{ae} R_{b}{}^{d} R^{bf} W_{cdef},$$
$$R_{c}{}^{e} R^{cg} R_{d}{}^{f} R^{dh} W^{ab}{}_{gh} W_{efab},\ -\frac{1}{4} R_{c}{}^{e} R^{ci} R_{d}{}^{g} R^{dj} W^{abfh} W_{egij} W_{fhab} + R_{e}{}^{g} R^{ei} R_{f}{}^{h} R^{fj} W^{ac}{}_{ij} W^{bd}{}_{ac} W_{ghbd} +$$
$$\frac{1}{12} R_{e}{}^{g} R^{e}{}_{i} R_{f}{}^{i} R^{f}{}_{g} W^{achj} W^{bd}{}_{ac} W_{hjbd} - \frac{1}{12} R_{e}{}^{g} R^{e}{}_{g} R_{f}{}^{i} R^{f}{}_{i} W^{achj} W^{bd}{}_{ac} W_{hjbd}\Big\}$$

8.2. Relation with our basis

The basis is given in terms of Weyl and Ricci. We need to convert the Weyl tensors into Riemann tensors, what takes quite a long time. The results form appendix B of the first paper.

Show timings over 1 second:

```
In[257]:=
      << xAct‘ShowTime1‘
```

The function do1 performs the translation from Weyl to Riemann (do1 has a very strange form in order to make it faster). The function do2 changes to invariants and simplify them:

```
In[258]:=
      do1[expr_] := ToCanonical@
         ContractMetric[ToCanonical@ContractMetric@WeylToRiemann@ToCanonical[expr] /.
           CurvatureRelations[CD]];
      do2[expr_] := expr // RiemannToInv // InvSimplify // Simplify;

In[260]:=
      $InvSimplifyLevel = 6

Out[260]=
      6
```

Pure Ricci:

```
In[261]:=
      I1 // do2
```

Out[261]=
$I_{0,1}$

```
In[262]:=
      I2 // do2
```

Out[262]=
$I_{00,1}$

```
In[263]:=
      I3 // do2
```

Out[263]=
$-I_{000,1}$

```
In[264]:=
      I4 // do2
```

Out[264]=
$I_{0000,1}$

Pure Weyl:

```
In[265]:=
      J1 // do1 // do2
```

Out[265]=
$$\frac{I_{0,1}^2}{3} - 2\, I_{00,1} + I_{00,2}$$

```
In[266]:=
      J2 // do1 // do1

          1.13607 Second

Out[266]=
```

$$6\,R_a{}^c\,R^{ab}\,R_{bc}-5\,R_{ab}\,R^{ab}\,R+\frac{5\,R^3}{9}+6\,R^{ab}\,R^{cd}\,R_{acbd}+R\,R_{abcd}\,R^{abcd}-6\,R^{ab}\,R_a{}^{cde}\,R_{bcde}+R_{ab}{}^{ef}\,R^{abcd}\,R_{cdef}$$

```
In[267]:=
      % // do2

Out[267]=
```

$$-\frac{17\,I_{0,1}^3}{18}+I_{0,1}\left(7\,I_{00,1}-\frac{I_{00,2}}{2}\right)+6\,I_{000,1}-6\,I_{000,2}+I_{000,5}$$

```
In[268]:=
      J3 // do1 // do1

          20.7773 Second

Out[268]=
```

$$-12\,R_a{}^c\,R^{ab}\,R_b{}^d\,R_{cd}+2\,R_{ab}\,R^{ab}\,R_{cd}\,R^{cd}+12\,R_a{}^c\,R^{ab}\,R_{bc}\,R-5\,R_{ab}\,R^{ab}\,R^2+\frac{5\,R^4}{12}+$$
$$8\,R^{ab}\,R^{cd}\,R\,R_{acbd}+\frac{1}{2}\,R^2\,R_{abcd}\,R^{abcd}-8\,R^{ab}\,R\,R_a{}^{cde}\,R_{bcde}-24\,R_a{}^c\,R^{ab}\,R^{de}\,R_{bdce}+$$
$$8\,R^{ab}\,R^{cd}\,R_{ac}{}^{ef}\,R_{bdef}+8\,R^{ab}\,R^{cd}\,R_a{}^e{}_c{}^f\,R_{bedf}+\frac{4}{3}\,R\,R_{ab}{}^{ef}\,R^{abcd}\,R_{cdef}+8\,R_a{}^c\,R^{ab}\,R_b{}^{def}\,R_{cdef}+$$
$$R_{ab}\,R^{ab}\,R_{cdef}\,R^{cdef}-8\,R^{ab}\,R_a{}^{cde}\,R_{bc}{}^{fg}\,R_{defg}+R_{ab}{}^{ef}\,R^{abcd}\,R_{cd}{}^{gh}\,R_{efgh}-\frac{1}{4}\,R_{abcd}\,R^{abcd}\,R_{efgh}\,R^{efgh}$$

```
In[269]:=
      % // do2

Out[269]=
```

$$-\frac{D_{00,2}^2}{16}$$

In[270]:=
```
J4 // do1 // do1
```

```
182.151 Second
```

Out[270]=

$$22\, R_{a}{}^{c}\, R^{ab}\, R_{b}{}^{d}\, R_{c}{}^{e}\, R_{de} - 5\, R_{ab}\, R^{ab}\, R_{c}{}^{e}\, R^{cd}\, R_{de} - 25\, R_{a}{}^{c}\, R^{ab}\, R_{b}{}^{d}\, R_{cd}\, R + \frac{5}{6}\, R_{ab}\, R^{ab}\, R_{cd}\, R^{cd}\, R + \frac{25}{2}\, R_{a}{}^{c}\, R^{ab}\, R_{bc}\, R^{2} - \frac{35}{12}\, R_{ab}\, R^{ab}\, R^{3} + \frac{7 R^{5}}{36} + \frac{35}{6}\, R^{ab}\, R^{cd}\, R^{2}\, R_{acbd} - \frac{35}{6}\, R^{ab}\, R^{2}\, R_{a}{}^{cde}\, R_{bcde} - 40\, R_{a}{}^{c}\, R^{ab}\, R^{de}\, R\, R_{bdce} + \frac{40}{3}\, R^{ab}\, R^{cd}\, R\, R_{ac}{}^{ef}\, R_{bdef} - 20\, R^{ab}\, R^{cd}\, R^{ef}\, R_{ace}{}^{g}\, R_{bdfg} + 30\, R_{a}{}^{c}\, R^{ab}\, R_{d}{}^{f}\, R^{de}\, R_{becf} + \frac{40}{3}\, R^{ab}\, R^{cd}\, R\, R_{a}{}^{e}{}_{c}{}^{f}\, R_{bedf} + \frac{35}{36}\, R^{2}\, R_{ab}{}^{ef}\, R^{abcd}\, R_{cdef} + \frac{40}{3}\, R_{a}{}^{c}\, R^{ab}\, R\, R_{b}{}^{def}\, R_{cdef} + 40\, R_{a}{}^{c}\, R^{ab}\, R_{b}{}^{d}\, R^{ef}\, R_{cedf} + 5\, R_{ab}\, R^{ab}\, R^{cd}\, R^{ef}\, R_{cedf} - 30\, R_{a}{}^{c}\, R^{ab}\, R^{de}\, R_{bd}{}^{fg}\, R_{cefg} - 20\, R_{a}{}^{c}\, R^{ab}\, R^{de}\, R_{b}{}^{f}{}_{d}{}^{g}\, R_{cfeg} + \frac{35}{12}\, R_{ab}\, R^{ab}\, R\, R_{cdef}\, R^{cdef} - \frac{40}{3}\, R^{ab}\, R\, R_{a}{}^{cde}\, R_{bc}{}^{fg}\, R_{defg} - 10\, R_{a}{}^{c}\, R^{ab}\, R_{b}{}^{d}\, R_{c}{}^{efg}\, R_{defg} - 5\, R_{ab}\, R^{ab}\, R^{cd}\, R_{c}{}^{efg}\, R_{defg} + 20\, R^{ab}\, R^{cd}\, R_{a}{}^{e}{}_{c}{}^{f}\, R_{be}{}^{gh}\, R_{dfgh} - \frac{5}{2}\, R_{a}{}^{c}\, R^{ab}\, R_{bc}\, R_{defg}\, R^{defg} + 10\, R^{ab}\, R^{cd}\, R_{ac}{}^{ef}\, R_{bd}{}^{gh}\, R_{efgh} + \frac{5}{3}\, R\, R_{ab}{}^{ef}\, R^{abcd}\, R_{cd}{}^{gh}\, R_{efgh} + 10\, R_{a}{}^{c}\, R^{ab}\, R_{b}{}^{def}\, R_{cd}{}^{gh}\, R_{efgh} + \frac{5}{6}\, R_{ab}\, R^{ab}\, R_{cd}{}^{gh}\, R^{cdef}\, R_{efgh} - \frac{5}{2}\, R^{ab}\, R^{cd}\, R_{acbd}\, R_{efgh}\, R^{efgh} - \frac{5}{12}\, R\, R_{abcd}\, R^{abcd}\, R_{efgh}\, R^{efgh} - 10\, R^{ab}\, R_{a}{}^{cde}\, R_{bc}{}^{fg}\, R_{de}{}^{hi}\, R_{fghi} + \frac{5}{2}\, R^{ab}\, R_{a}{}^{cde}\, R_{bcde}\, R_{fghi}\, R^{fghi} + R_{ab}{}^{ef}\, R^{abcd}\, R_{cd}{}^{gh}\, R_{ef}{}^{ij}\, R_{ghij} - \frac{5}{12}\, R_{abcd}\, R^{abcd}\, R_{ef}{}^{ij}\, R^{efgh}\, R_{ghij}$$

In[271]:=
```
% // do2
```

Out[271]=

$$\frac{5}{96}\, D_{00,2}\, (-4\, D_{000,2} - 2\, D_{000,13} + D_{00,2}\, I_{0,1})$$

Mixed invariants:

In[272]:=
```
K1 // do1 // do1
```

Out[272]=

$$R_{a}{}^{c}\, R^{ab}\, R_{bc} - \frac{7}{6}\, R_{ab}\, R^{ab}\, R + \frac{R^{3}}{6} + R^{ab}\, R^{cd}\, R_{acbd}$$

In[273]:=
```
% // do2
```

Out[273]=

$$\frac{1}{6}\, (I_{0,1}^{3} - 7\, I_{0,1}\, I_{00,1} - 6\, I_{000,1} + 6\, I_{000,2})$$

In[274]:=
```
K2 // do1 // do1
```

Out[274]=

$$-2\, R_{a}{}^{c}\, R^{ab}\, R_{b}{}^{d}\, R_{cd} + R_{ab}\, R^{ab}\, R_{cd}\, R^{cd} + \frac{5}{3}\, R_{a}{}^{c}\, R^{ab}\, R_{bc}\, R - \frac{13}{18}\, R_{ab}\, R^{ab}\, R^{2} + \frac{R^{4}}{18} + \frac{2}{3}\, R^{ab}\, R^{cd}\, R\, R_{acbd} - 4\, R_{a}{}^{c}\, R^{ab}\, R^{de}\, R_{bdce} + R^{ab}\, R^{cd}\, R_{ac}{}^{ef}\, R_{bdef}$$

In[275]:=

```
% // do2
```

Out[275]=

$$-\frac{5\, I_{0,1}^4}{18} + \frac{41}{18} I_{0,1}^2\, I_{00,1} - I_{00,1}^2 + I_{0,1}\left(3\, I_{000,1} - \frac{4\, I_{000,2}}{3}\right) + 2\, I_{0000,1} + I_{0000,5}$$

In[276]:=

```
K3 // do1 // do1
```

```
8.20851 Second
```

Out[276]=

$$\begin{aligned}
&4\, {R_a}^c R^{ab} {R_b}^d {R_c}^e R_{de} - 2\, R_{ab} R^{ab} {R_c}^e R^{cd} R_{de} - 3\, {R_a}^c R^{ab} {R_b}^d R_{cd}\, R + \frac{3}{4} {R_a}^c R^{ab} R_{bc}\, R^2 + \\
&\frac{7}{24} R_{ab} R^{ab} R^3 - \frac{R^5}{24} - \frac{1}{4} R^{ab} R^{cd} R^2 R_{acbd} - \frac{1}{8} R^3 R_{abcd} R^{abcd} + \frac{1}{2} R^{ab} R^2 {R_a}^{cde} R_{bcde} - \\
&4\, {R_a}^c R^{ab} R^{de} R\, R_{bdce} + R^{ab} R^{cd} R\, {R_{ac}}^{ef} R_{bdef} - 2\, R^{ab} R^{cd} R^{ef} {R_{ace}}^g R_{bdfg} + 6\, {R_a}^c R^{ab} {R_d}^f R^{de} R_{becf} - \\
&\frac{1}{12} R^2 {R_{ab}}^{ef} R^{abcd} R_{cdef} + 6\, {R_a}^c R^{ab} {R_b}^d R^{ef} R_{cedf} + R_{ab} R^{ab} R^{cd} R^{ef} R_{cedf} - 4\, {R_a}^c R^{ab} R^{de} {R_{bd}}^{fg} R_{cefg} + \\
&\frac{3}{8} R_{ab} R^{ab} R\, R_{cdef} R^{cdef} - \frac{1}{2} R_{ab} R^{ab} R^{cd} {R_c}^{efg} R_{defg} - \frac{1}{4} {R_a}^c R^{ab} R_{bc} R_{defg} R^{defg} + \\
&R^{ab} R^{cd} {R_{ac}}^{ef} {R_{bd}}^{gh} R_{efgh} + \frac{1}{12} R_{ab} R^{ab} {R_{cd}}^{gh} R^{cdef} R_{efgh} - \frac{1}{4} R^{ab} R^{cd} R_{acbd} R_{efgh} R^{efgh}
\end{aligned}$$

In[277]:=

```
% // do2
```

Out[277]=

$$\frac{D_{00,2}\, D_{000,2}}{16}$$

In[278]:=

```
K4 // do1 // do1
```

Out[278]=

$$\begin{aligned}
&{R_a}^c R^{ab} {R_b}^d {R_c}^e R_{de} - R_{ab} R^{ab} {R_c}^e R^{cd} R_{de} - \\
&\frac{1}{6} {R_a}^c R^{ab} {R_b}^d R_{cd}\, R + \frac{1}{6} R_{ab} R^{ab} R_{cd} R^{cd} R + {R_a}^c R^{ab} {R_d}^f R^{de} R_{becf}
\end{aligned}$$

In[279]:=

```
% // do2
```

Out[279]=

$$\begin{aligned}
&\frac{1}{6} \big(I_{0,1}^5 - 8\, I_{0,1}^3\, I_{00,1} + I_{0,1}^2\, (-11\, I_{000,1} + 3\, I_{000,2}) + \\
&\quad I_{00,1}\, (I_{000,1} + 3\, I_{000,2}) + I_{0,1}\, (4\, I_{00,1}^2 - 7\, I_{0000,1}) - 12\, I_{00000,2}\big)
\end{aligned}$$

In[280]:=

```
K5 // do1 // do1
```

Out[280]=

$$\begin{aligned}
&-2\, {R_a}^c R^{ab} {R_b}^d {R_c}^e {R_d}^f R_{ef} + R_{ab} R^{ab} {R_c}^e R^{cd} {R_d}^f R_{ef} + {R_a}^c R^{ab} R_{bc} {R_d}^f R^{de} R_{ef} + \frac{2}{3} {R_a}^c R^{ab} {R_b}^d {R_c}^e R_{de}\, R - \\
&\frac{2}{3} R_{ab} R^{ab} {R_c}^e R^{cd} R_{de}\, R - \frac{1}{18} {R_a}^c R^{ab} {R_b}^d R_{cd}\, R^2 + \frac{1}{18} R_{ab} R^{ab} R_{cd} R^{cd} R^2 + \\
&\frac{2}{3} {R_a}^c R^{ab} {R_d}^f R^{de} R\, R_{becf} - 4\, {R_a}^c R^{ab} {R_b}^d {R_e}^g R^{ef} R_{cfdg} + {R_a}^c R^{ab} {R_d}^f R^{de} {R_{be}}^{gh} R_{cfgh}
\end{aligned}$$

In[281]:=
% // do2

Out[281]=

$$\begin{aligned}&\frac{1}{72}\,(2\,I_{0,1}^{6}-I_{0,1}^{4}\,(16\,I_{00,1}+3\,I_{00,2})-\\
&\quad 4\,I_{0,1}^{3}\,(13\,I_{000,1}+6\,I_{000,2})+4\,I_{0,1}^{2}\,(7\,I_{00,1}^{2}-13\,I_{0000,1}+9\,I_{0000,5})+\\
&\quad 4\,I_{0,1}\,(2\,I_{00,1}\,(7\,I_{000,1}+3\,I_{000,2})-3\,(I_{00,2}\,I_{000,1}+8\,I_{00000,2}))-3\,(6\,I_{00,1}^{3}-3\,I_{00,1}^{2}\,I_{00,2}-\\
&\qquad 8\,I_{000,1}^{2}+16\,I_{000,1}\,I_{000,2}+6\,I_{00,2}\,I_{0000,1}-12\,I_{00,1}\,(I_{0000,1}+I_{0000,5})+48\,I_{000000,6}))\end{aligned}$$

In[282]:=
K6 // do1 // do1

12.3368 Second

Out[282]=

$$\begin{aligned}&4\,R_a{}^c\,R^{ab}\,R_b{}^d\,R_c{}^e\,R_d{}^f\,R_e{}^g\,R_{fg}-\frac{1}{2}\,R_{ab}\,R^{ab}\,R_c{}^e\,R^{cd}\,R_d{}^f\,R_e{}^g\,R_{fg}-\frac{5}{2}\,R_a{}^c\,R^{ab}\,R_{bc}\,R_d{}^f\,R^{de}\,R_e{}^g\,R_{fg}-\\
&R_{ab}\,R^{ab}\,R_{cd}\,R^{cd}\,R_e{}^g\,R^{ef}\,R_{fg}-2\,R_a{}^c\,R^{ab}\,R_b{}^d\,R_c{}^e\,R_d{}^f\,R_{ef}\,R+\frac{1}{2}\,R_{ab}\,R^{ab}\,R_c{}^e\,R^{cd}\,R_d{}^f\,R_{ef}\,R+\\
&R_a{}^c\,R^{ab}\,R_{bc}\,R_d{}^f\,R^{de}\,R_{ef}\,R+\frac{1}{2}\,R_{ab}\,R^{ab}\,R_{cd}\,R^{cd}\,R_{ef}\,R^{ef}\,R+\frac{1}{4}\,R_a{}^c\,R^{ab}\,R_b{}^d\,R_c{}^e\,R_{de}\,R^2-\\
&\frac{1}{4}\,R_{ab}\,R^{ab}\,R_c{}^e\,R^{cd}\,R_{de}\,R^2+\frac{1}{24}\,R_a{}^c\,R^{ab}\,R_b{}^d\,R_{cd}\,R^3-\frac{1}{24}\,R_{ab}\,R^{ab}\,R_{cd}\,R^{cd}\,R^3+\\
&\frac{1}{4}\,R_a{}^c\,R^{ab}\,R_d{}^f\,R^{de}\,R^2\,R_{becf}-4\,R_a{}^c\,R^{ab}\,R_b{}^d\,R_e{}^g\,R^{ef}\,R\,R_{cfdg}+R_a{}^c\,R^{ab}\,R_d{}^f\,R^{de}\,R\,R_{be}{}^{gh}\,R_{cfgh}-\\
&2\,R_a{}^c\,R^{ab}\,R_d{}^f\,R^{de}\,R^{gh}\,R_{beg}{}^i\,R_{cfhi}+6\,R_a{}^c\,R^{ab}\,R_b{}^d\,R_e{}^g\,R^{ef}\,R_f{}^h\,R_{cgdh}+6\,R_a{}^c\,R^{ab}\,R_b{}^d\,R_c{}^e\,R_f{}^h\,R^{fg}\,R_{dgeh}+\\
&\frac{1}{2}\,R_{ab}\,R^{ab}\,R_c{}^e\,R^{cd}\,R_f{}^h\,R^{fg}\,R_{dgeh}-4\,R_a{}^c\,R^{ab}\,R_b{}^d\,R_e{}^g\,R^{ef}\,R_{cf}{}^{hi}\,R_{dghi}+\frac{1}{2}\,R_a{}^c\,R^{ab}\,R_b{}^d\,R_{cd}\,R^{ef}\,R^{gh}\,R_{egfh}-\\
&\frac{1}{2}\,R_{ab}\,R^{ab}\,R_{cd}\,R^{cd}\,R^{ef}\,R^{gh}\,R_{egfh}+\frac{1}{8}\,R_a{}^c\,R^{ab}\,R_b{}^d\,R_{cd}\,R\,R_{efgh}\,R^{efgh}-\frac{1}{8}\,R_{ab}\,R^{ab}\,R_{cd}\,R^{cd}\,R\,R_{efgh}\,R^{efgh}-\\
&\frac{1}{2}\,R_a{}^c\,R^{ab}\,R_b{}^d\,R_{cd}\,R^{ef}\,R_e{}^{ghi}\,R_{fghi}+\frac{1}{2}\,R_{ab}\,R^{ab}\,R_{cd}\,R^{cd}\,R^{ef}\,R_e{}^{ghi}\,R_{fghi}-\\
&\frac{1}{4}\,R_a{}^c\,R^{ab}\,R_b{}^d\,R_c{}^e\,R_{de}\,R_{fghi}\,R^{fghi}+\frac{1}{4}\,R_{ab}\,R^{ab}\,R_c{}^e\,R^{cd}\,R_{de}\,R_{fghi}\,R^{fghi}+\\
&R_a{}^c\,R^{ab}\,R_d{}^f\,R^{de}\,R_{be}{}^{gh}\,R_{cf}{}^{ij}\,R_{ghij}+\frac{1}{12}\,R_a{}^c\,R^{ab}\,R_b{}^d\,R_{cd}\,R_{ef}{}^{ij}\,R^{efgh}\,R_{ghij}-\\
&\frac{1}{12}\,R_{ab}\,R^{ab}\,R_{cd}\,R^{cd}\,R_{ef}{}^{ij}\,R^{efgh}\,R_{ghij}-\frac{1}{4}\,R_a{}^c\,R^{ab}\,R_d{}^f\,R^{de}\,R_{becf}\,R_{ghij}\,R^{ghij}\end{aligned}$$

In[283]:=
% // do2

1.16007 Second

Out[283]=

$$\frac{1}{32}\,D_{00,2}\,(-4\,D_{00000,2}+D_{000,2}\,(I_{0,1}^{2}+I_{00,1}))$$

■ 9. Final comments

9.1. ToDo list. Version

There are many things that `Invar‘` cannot do yet. Here there is a list with some of them:

− Weyl invariants.

− Free indices.

− Dimensionally−dependent identities for dimensions other than 4.

Note: For further information about `Invar‘`, and to be kept informed about new releases, you may contact the authors electronically at jmm@iem.cfmac.csic.es, yllanes@lattice.fis.ucm.es. This is InvarDoc.nb, the docfile of Invar, currently in version 2.0. This document contains entirely the doc file of version 1.0 of Invar.

9.2. Frequently Asked Questions

None yet.

9.3. Statistics

Collection of public symbols in `Invar‘`:

```
In[284]:=
     Names["xAct‘Invar‘*"]

Out[284]=
     {dim, Disclaimer, DualRInv, DualRInvRules, DualRInvs, DualWInv,
      DualWInvRules, DualWInvs, InvarCases, InvarDirectory, InvarDualCases,
      InvSimplify, InvToPerm, InvToRiemann, MaxDualIndex, MaxIndex, PermToInv,
      PermToRiemann, RandomRiemannMonomial, RemoveDualRInvRules, RemoveRInvRules,
      RiemannSimplify, RiemannToInv, RiemannToPerm, RInv, RInvRules, RInvs, RPerm,
      RToW, RToWDualRules, RToWRules, sigma, WInv, WInvRules, WInvs, WPerm, WToR,
      WToRDualRules, WToRRules, $CurvatureRelations, $ExpandedCommuteOrder12Q,
      $InvarDirectory, $InvSimplifyLevel, $Version, $xTensorVersionExpected}

In[285]:=
     Length[%]

Out[285]=
     45

In[286]:=
     TimeUsed[]

Out[286]=
     278.021

In[287]:=
     MemoryInUse[] / 1000 / 1024 // N

Out[287]=
     464.523
```

```
In[288]:=
     MaxMemoryUsed[] / 1000 / 1024 // N

Out[288]=
     484.799
```