xAct

Efficient manipulation of tensor expressions

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Computers in science

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- Numerics: Approximate solutions to continuous problems.
- Computers are discrete-calculus machines!
- Computer algebra (CA): Exact solutions.
- Our problem: Efficient Tensor Computer Algebra (TCA).

Summary

1. General purpose computer algebra (CA)
   - Focus: the canonicalizer.
2. Computer algebra for tensor calculus (TCA)
   - Focus: types of problems.
3. A *Mathematica* / C implementation: *xAct*
4. Case example: scalars of the Riemann tensor

1. Computer algebra (CA)

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  - No truncation of information
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- Early history:
  - 1950: ALGAE (Los Alamos)
  - 1953: Kahrimanian, Nolan: differentiation systems
  - 1963: ALTRAN / ALPAK (Bell Labs)
  - 1960’s: LISP: recursion, conditionals, dynamical allocation of memory, garbage collection, etc.

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- Specialised systems for:
  - Celestial mechanics
  - Group theory: MAGMA, GAP
  - General Relativity
  - Quantum Field Theory
  - Fluid mechanics
  - Industrial applications

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Examples:

- $\det(A+B) \quad \longrightarrow \quad n!\, 2^n$ terms $\;(\simeq 4 \cdot 10^9$ for $n = 10\,)$

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Examples:

- *Intermediate expression swell*:
  Simple input $\longrightarrow$ Complex intermediate steps $\longrightarrow$ Simple output

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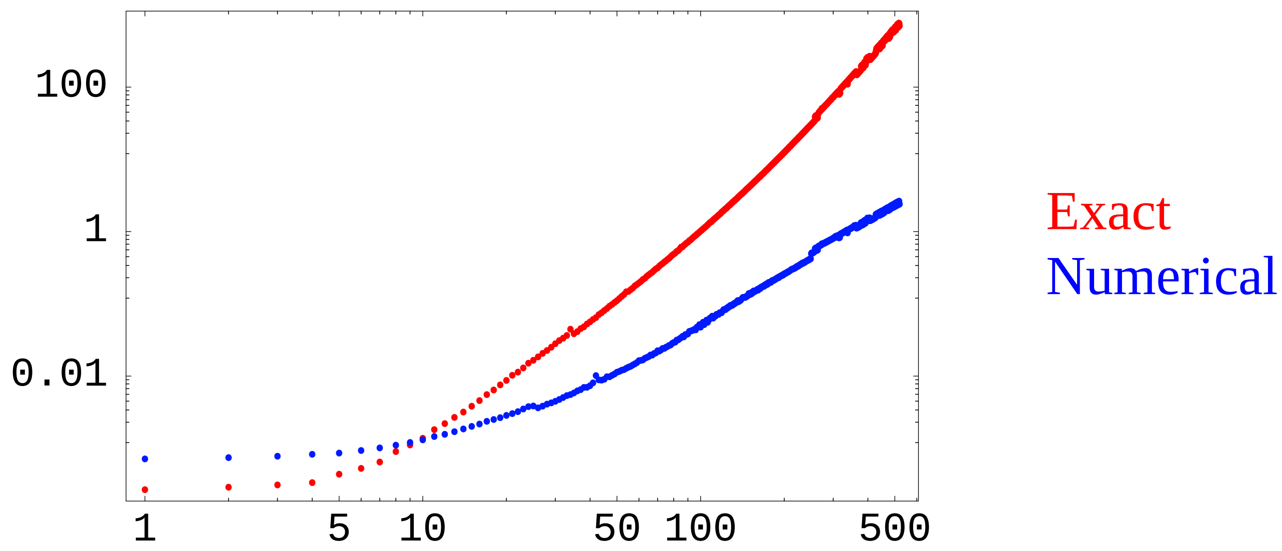
Examples:

- Linear systems with random integers $|c_i| \leq 100$. Timing (s):



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Is Computer Algebra doomed to failure?

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$$\texttt{Simplify}[\,(x-1)(x+1)\,] \quad \longrightarrow \quad -1+x^2$$
$$\texttt{Simplify}[\,(x-y)(x+y)\,] \quad \longrightarrow \quad (x-y)(x+y)$$
$$\texttt{Simplify}[\,x^4-x\,] \quad \longrightarrow \quad x(-1+x^3)$$

2. Computer algebra for (GR) tensors

Motivation: Long but “simple” problems

- Perturbation theory
- Bel-Robinson, super energy-momentum tensors, ...

$$4B_{abcd} = C_a{}^e{}_b{}^f C_{cedf} + {}^*C_a{}^e{}_b{}^{f\,*}C_{cedf} \quad \Rightarrow \quad B_{abcd} = B_{(abcd)}$$

- Riemann polynomials:

$$R_{abcd}R^a{}_e{}^c{}_f R^{bfde} = R_{abcd}R^a{}_e{}^c{}_f R^{bedf} - \frac{1}{4}R_{abcd}R^{ab}{}_{ef}R^{cdef}$$

- Lovelock (dimension-dependent) identities
- Component expansions in numerics (code generation)
  `[ Kranc:  Husa, Hinder, Lechner ’04 ]`
- Manipulation and classification of metrics
  `[ GRDB: Ishak, Lake ’02; ICD: Skea ’97]`
- Added benefits: reproducibility, error-free, ...

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R. d'Inverno 1969

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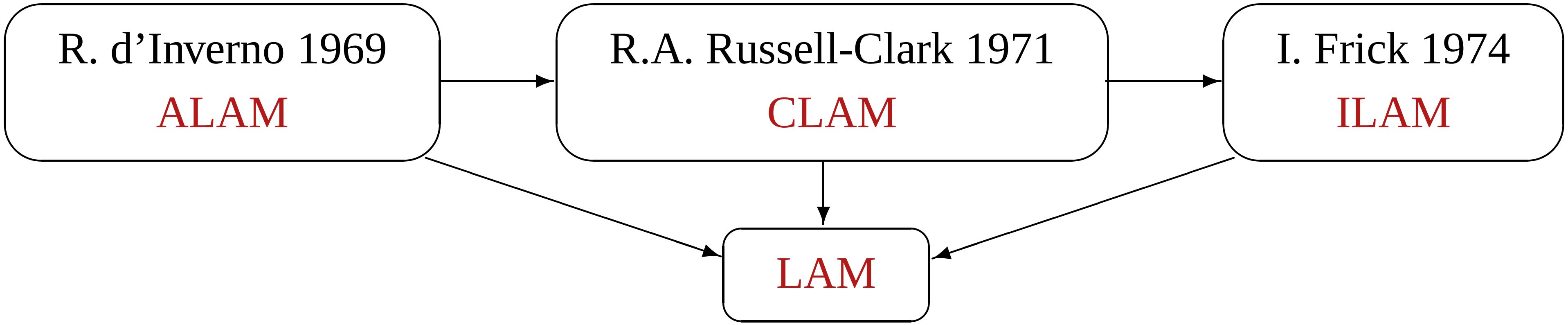

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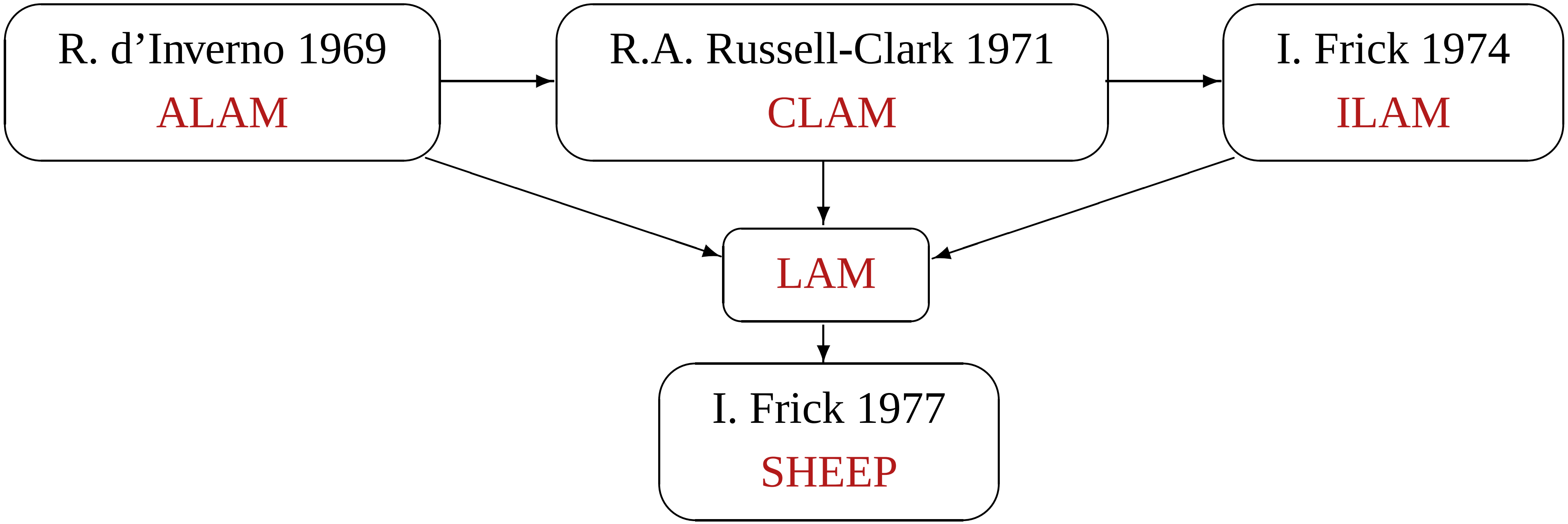

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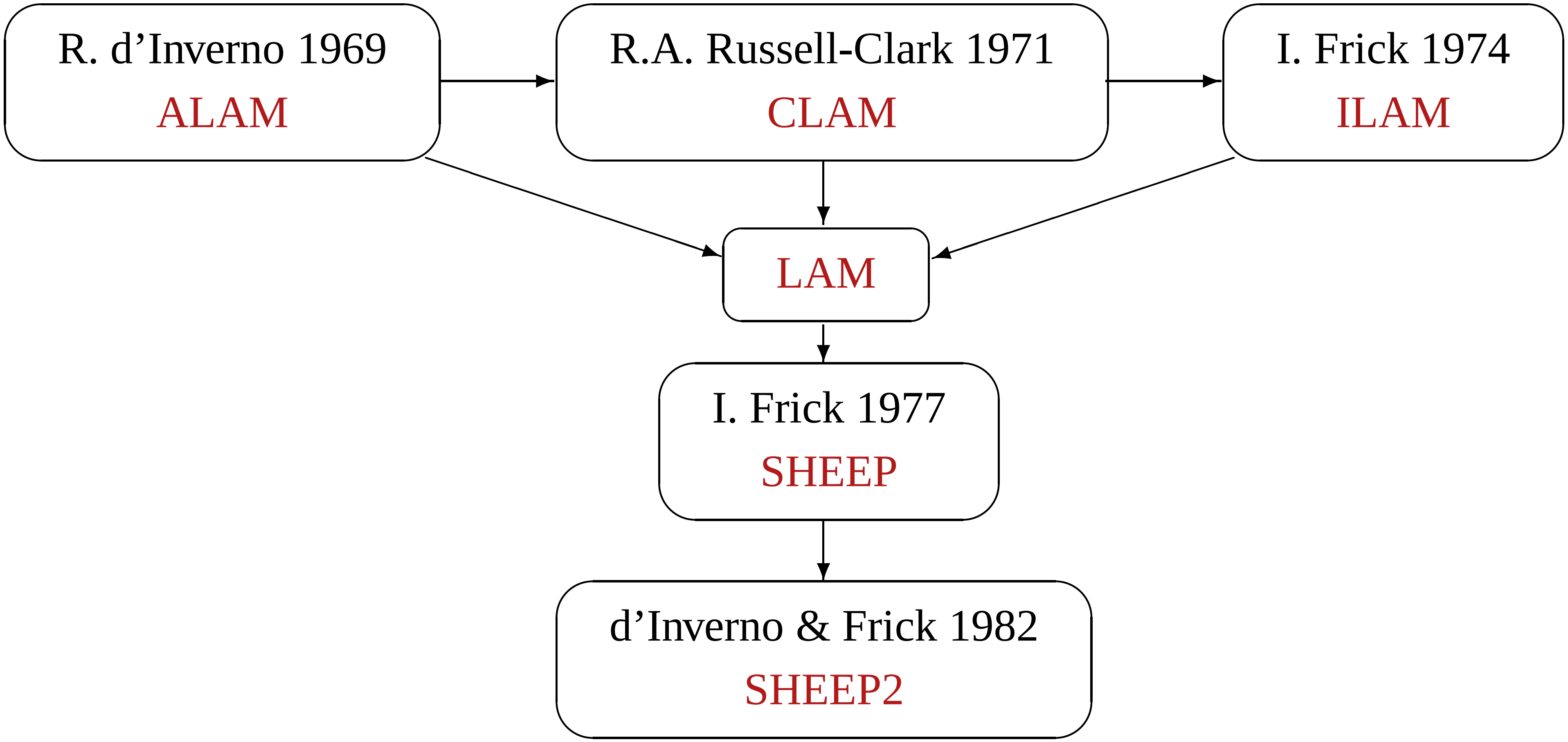

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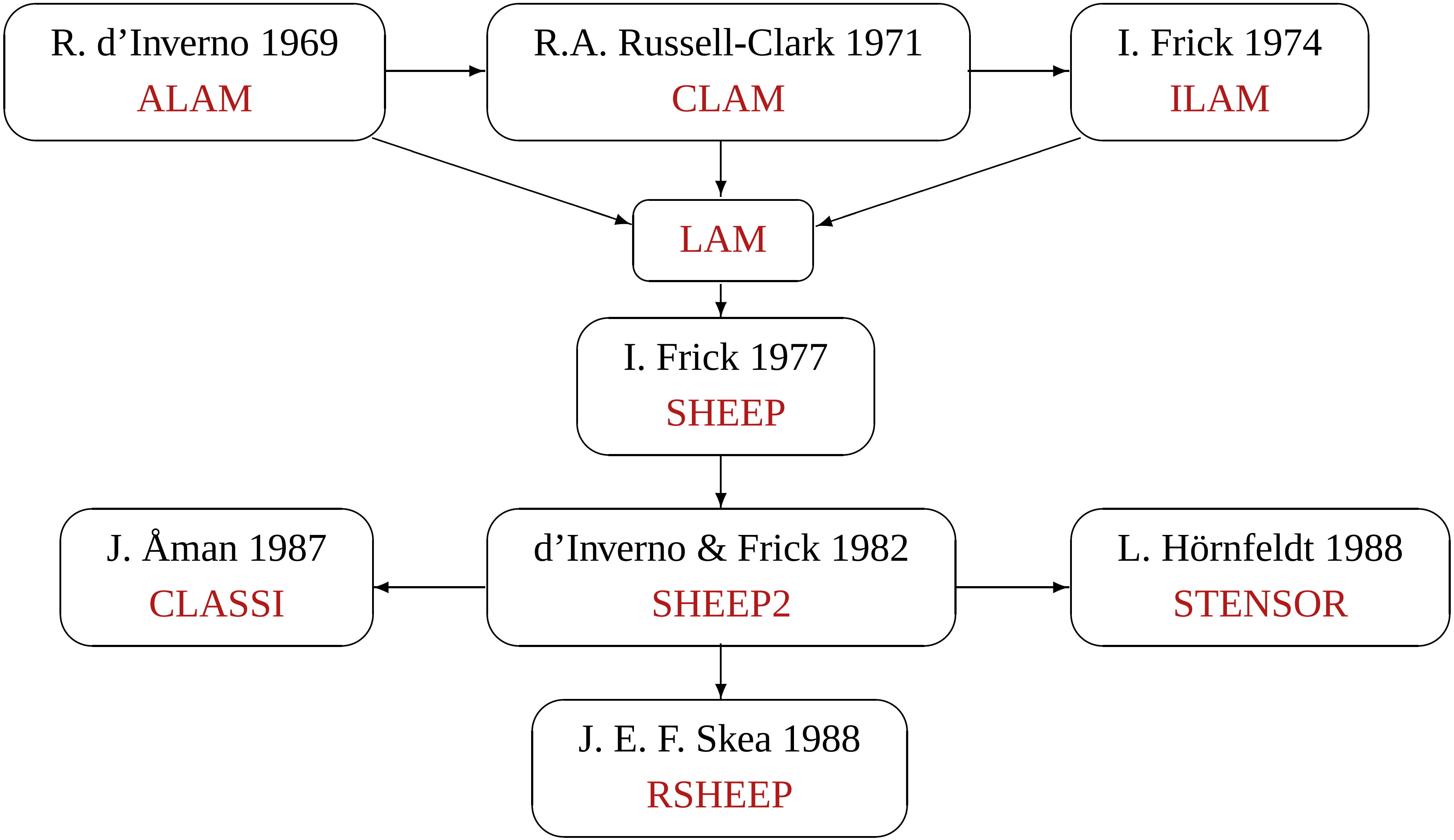

2c. Classes: types of problems

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Component computations:

- Give a metric in a coordinate system / frame.
- Compute other tensors from that metric.
- Key issues:
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Question: Component computations from abstract computations?

2c-2. Classes: sources of complexity

- General expression: tensor polynomial

$$\ldots + 3r^2 R_{abcd} R^{aecf} T^b{}_e \nabla_f \, v^d + R^{abcd} R_{cdef} R^{ef}{}_{ab} + \ldots$$

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- Arrange dummy indices in full expression.

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Monoterm symmetries (perm groups):

$$R_{bacd} = -R_{abcd}, \qquad R_{cdab} = +R_{abcd}$$

- Most packages use ad hoc exponential algorithms.
- Polynomic algorithms to manipulate the Symmetric Group S_n, based on Strong Generating Set representations
  `[ Schreier, Sims, Knuth, ... 70's, 80's ].`
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Multiterm symmetries (perm algebras):

$$R_{abcd} + R_{acdb} + R_{adbc} = 0$$

- No known efficient algorithms. Solutions?
- Most elegant: Young tableaux `[ Fulling et al. '92; Peeters '05 ]`
- Particular case: dimension-dependent identities `[ Edgar et al. '02 ].`

2d. Tensor packages

MAXIMA: itensor, stensor / ctensor

REDUCE: atensor, RicciR, ExCalc / GRG, GRLIB, RedTen

MAPLE: Riegeom, Canon, MapleTensor / Riemann, Atlas, GRTensorII

MATHEMATICA: MathTensor ($$), dhPark, Tensors in Physics ($), Tensorial ($), Ricci, TTC, EinS, xTensor / GRTensorM, xCoba

Standalone: cadabra

Prototyping: Kranc, RNPL, TeLa

Many other small packages for component computations.

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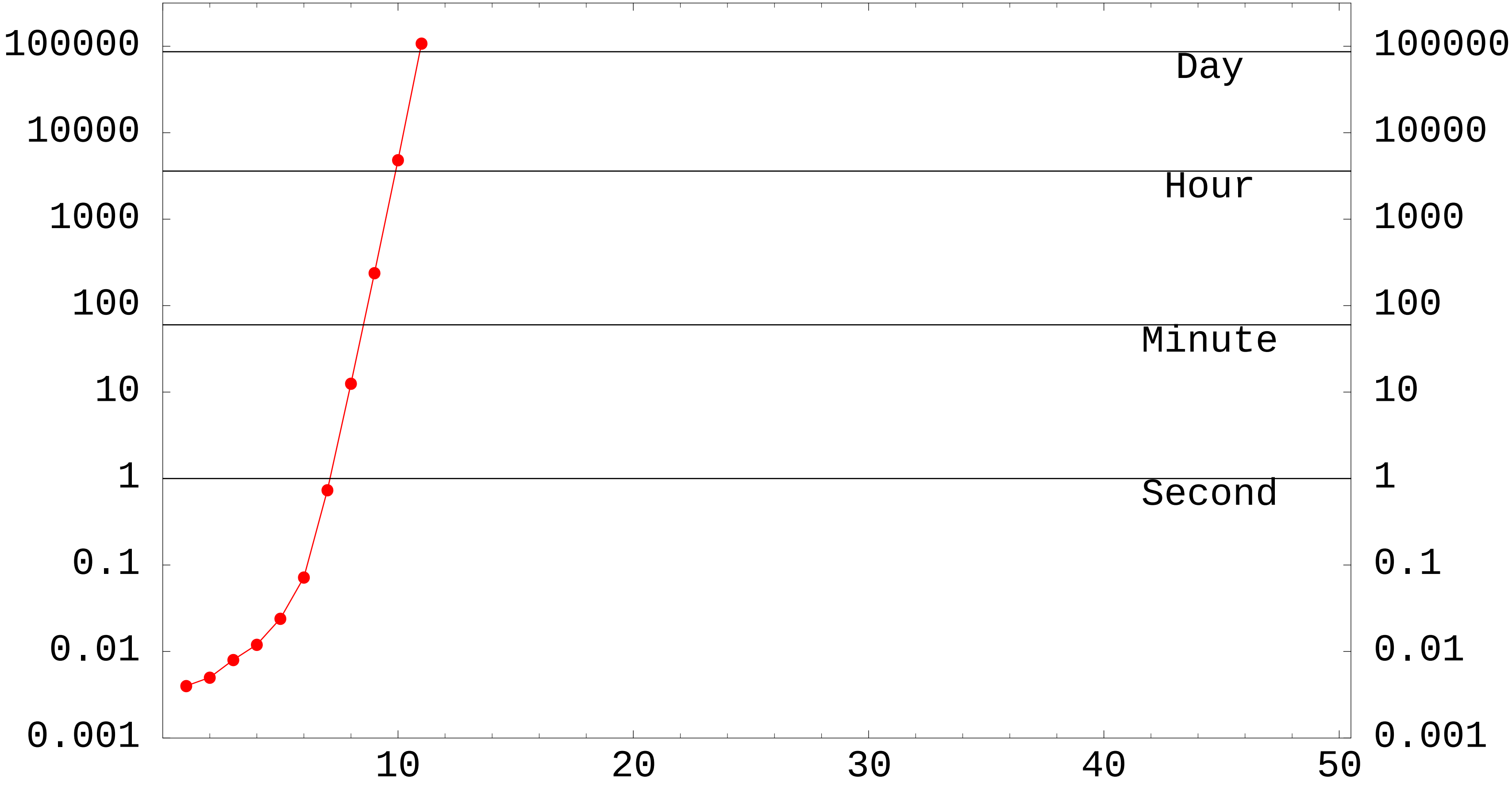
- $F^{ab} F_{bc} \overset{n}{\cdots} F^{h}{}_{a} = 0$ if number n of tensors is odd.
- Timings (in seconds)

n	|Perm group|	MathTensor	xTensor
1	2	0	0
3	48	0.01	0.01
5	3840	0.02	0.03
7	645120	1.12	0.05
9	185794560	350	0.07
11	$8.2\ 10^{10}$	107745	0.09
19	$6.4\ 10^{22}$	?	0.28
29	$4.7\ 10^{39}$	?	0.94
39	$1.1\ 10^{58}$	?	2.7
49	$3.4\ 10^{77}$	?	6.5
59	$8.0\ 10^{97}$	?	13.7

2e-2. Example 1



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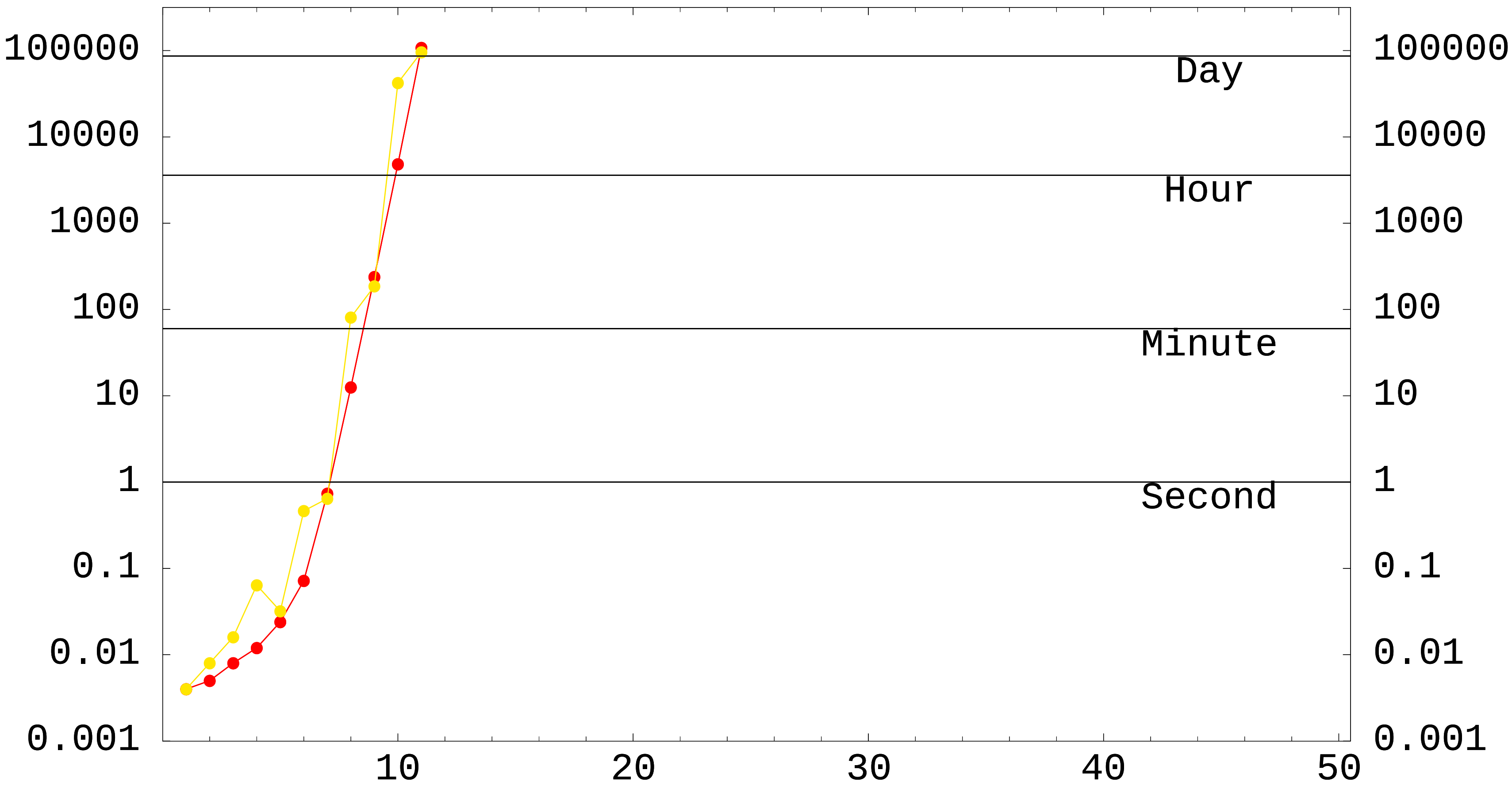

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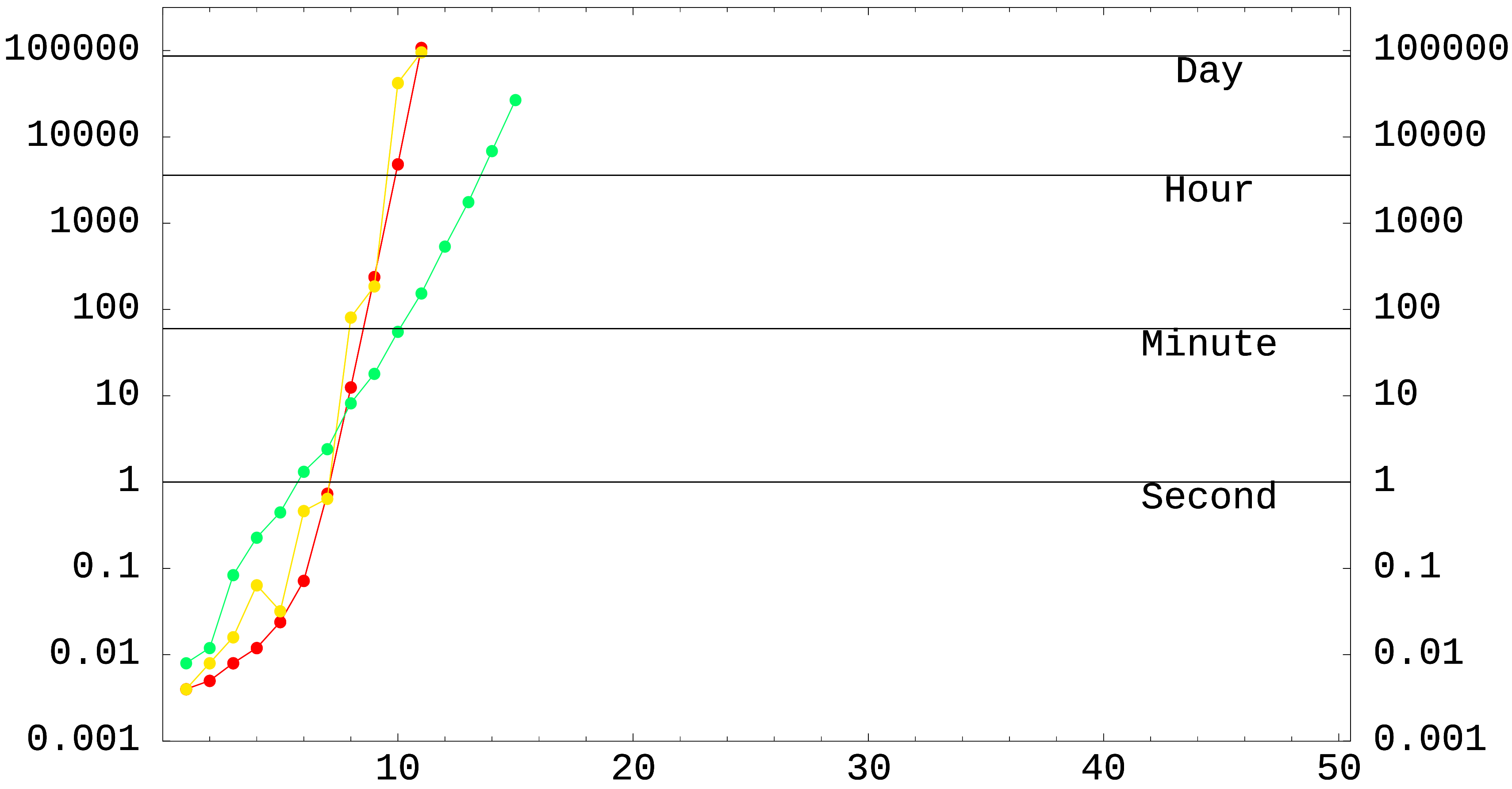

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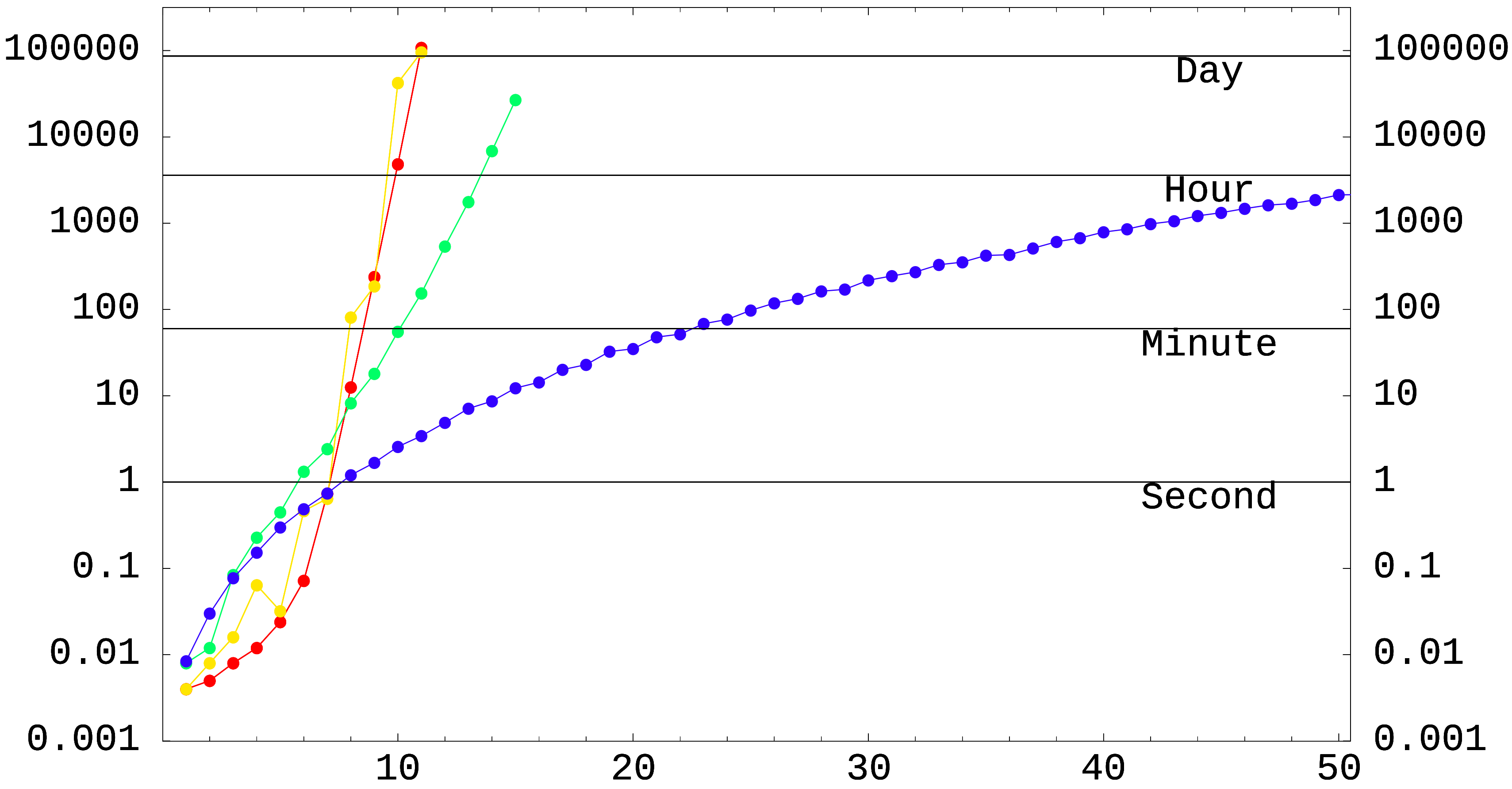

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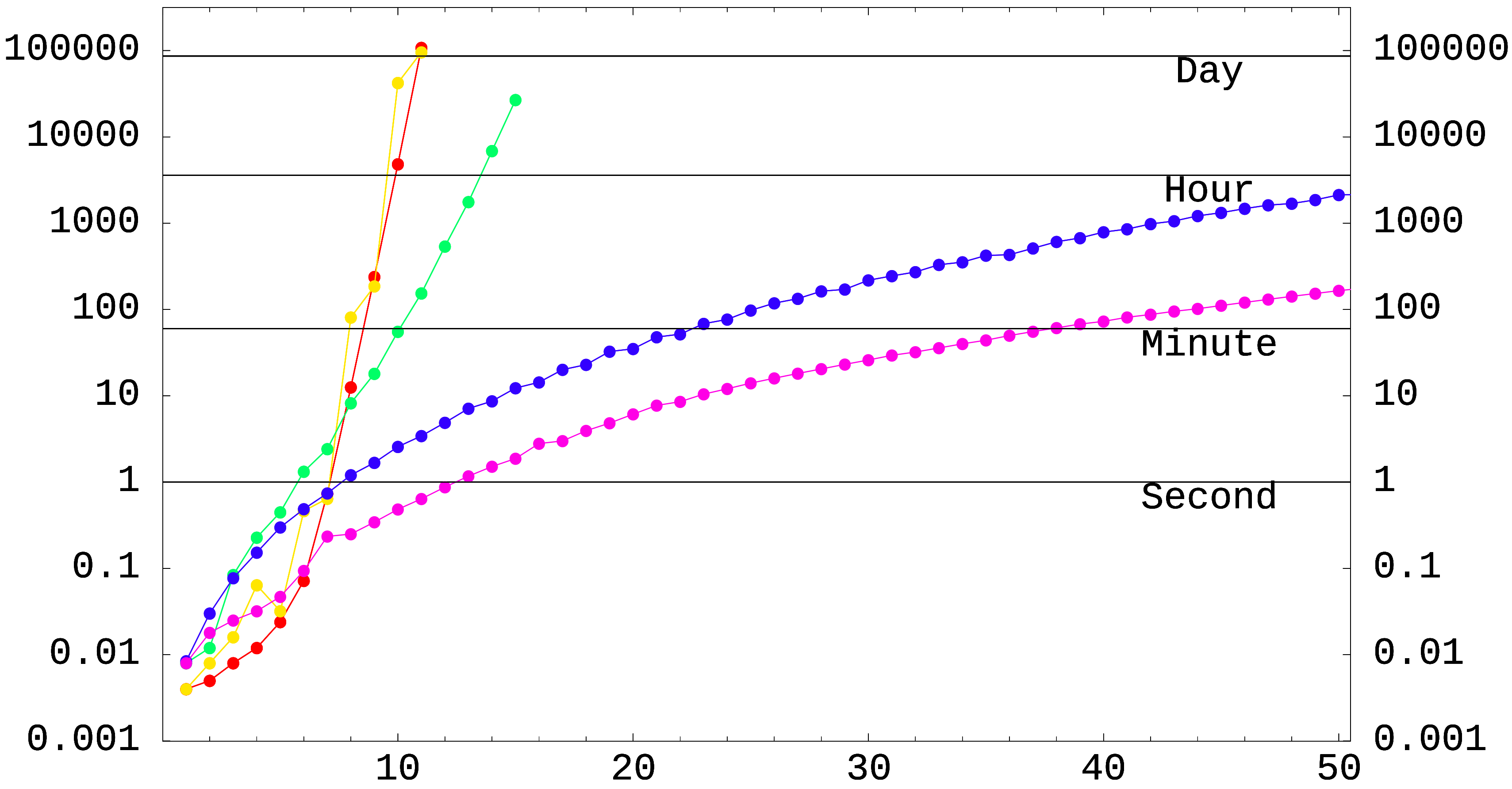

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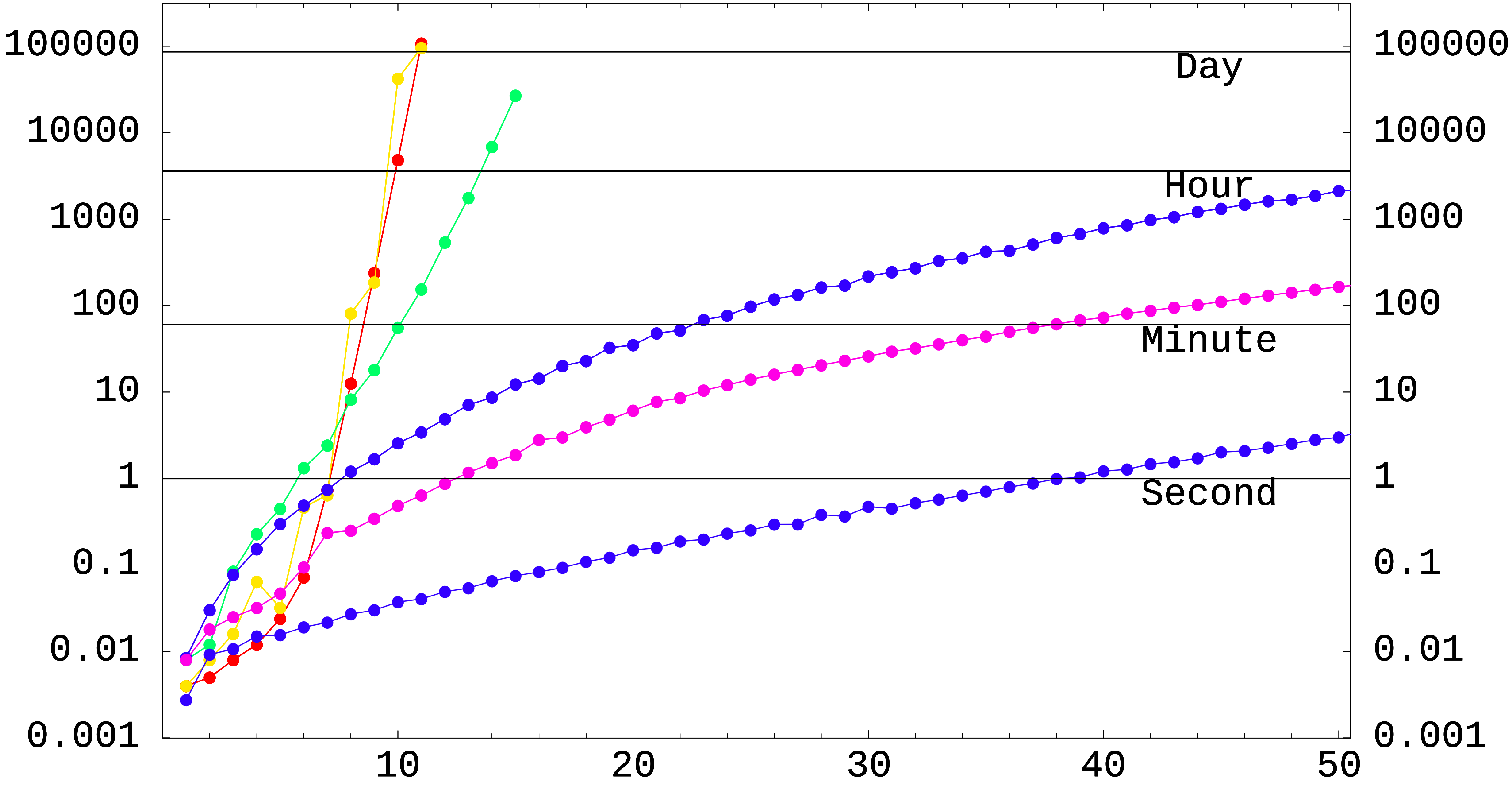
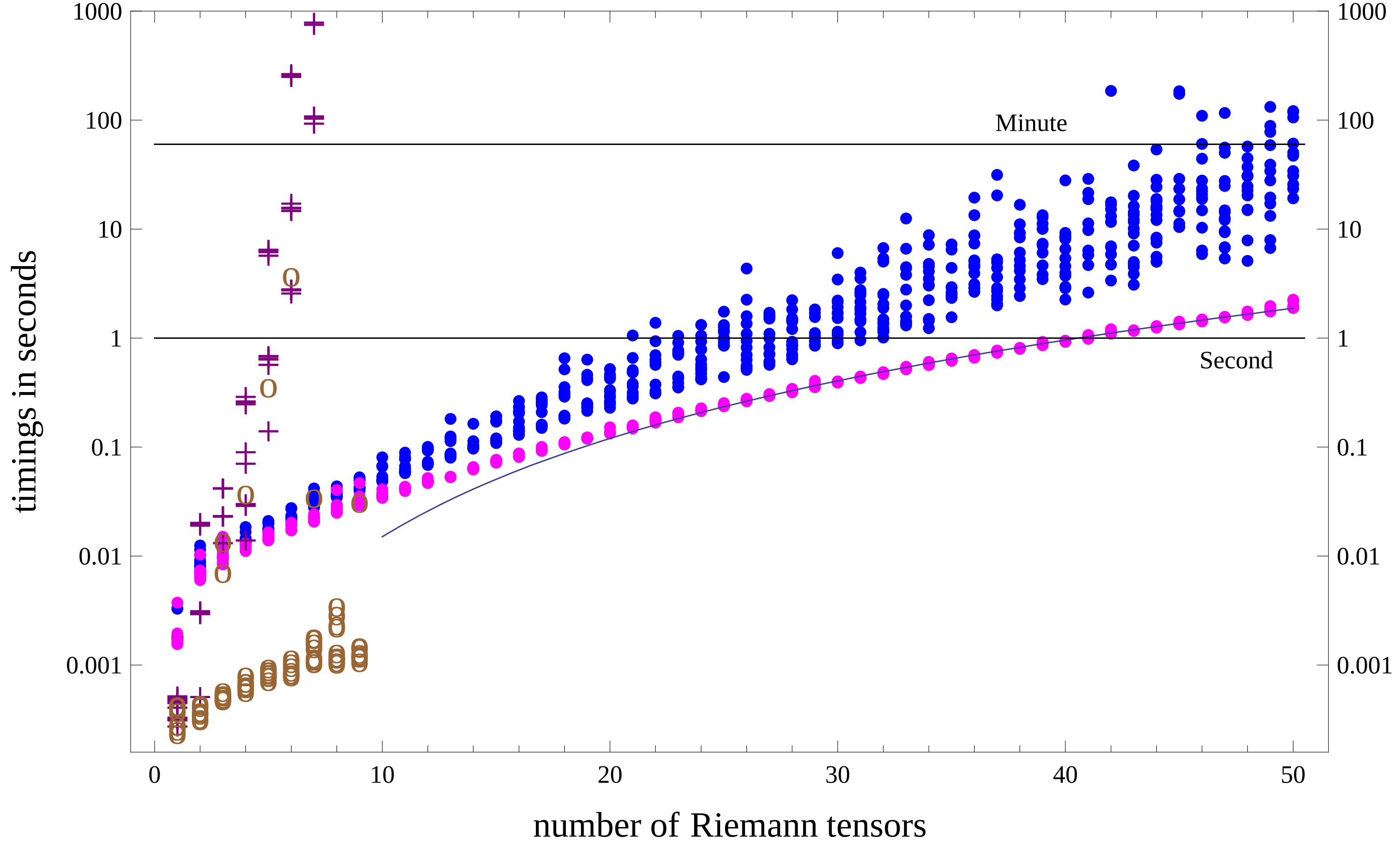


2e-3. Example 2

Random monomial Riemann scalars:

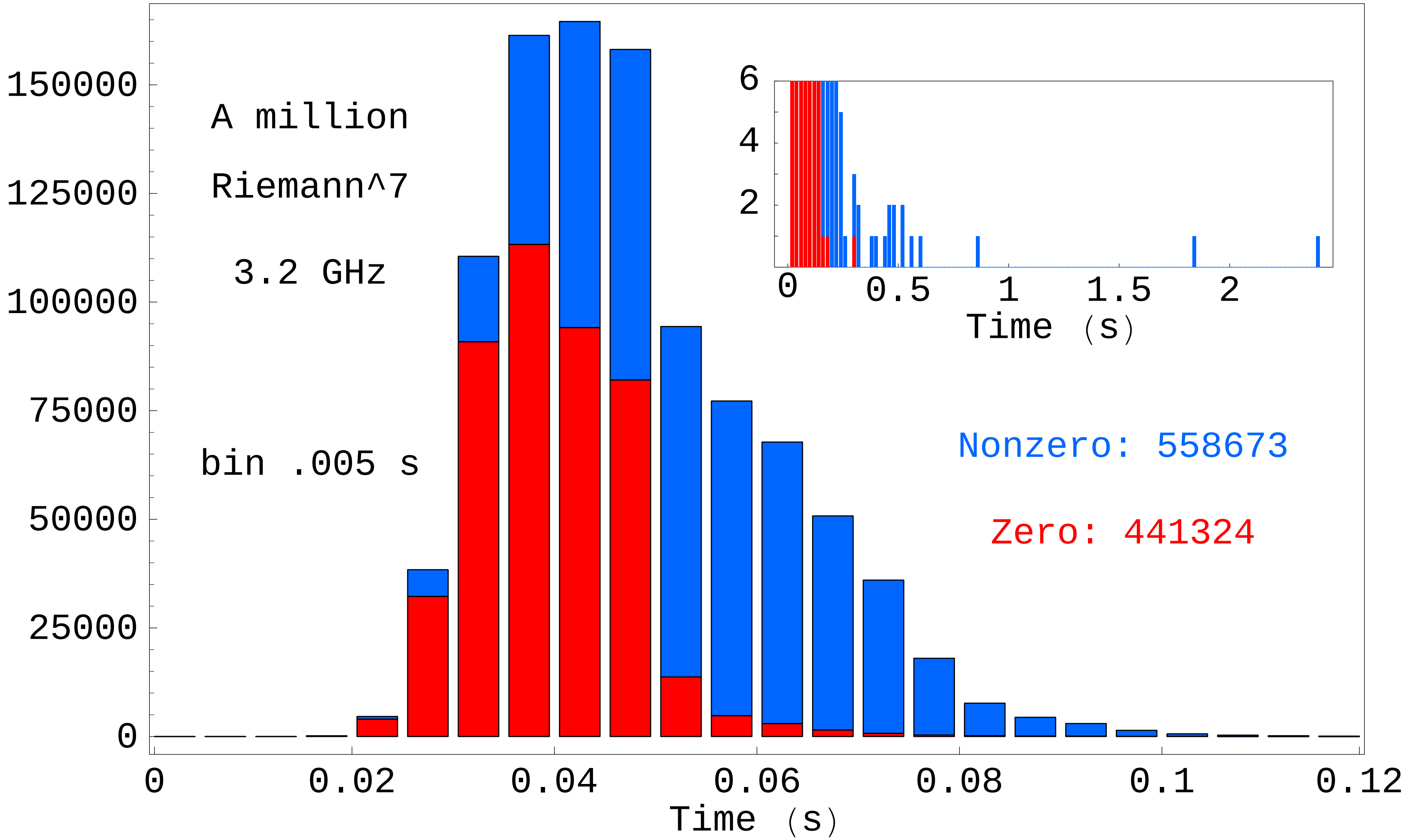
$$g^{a_1 b_7} \dots g^{d_n c_5} R_{a_1 b_1 c_1 d_1} \dots R_{a_n b_n c_n d_n}$$



The algorithm uses the intersection algorithm, which is known to be exponential in the worst case.

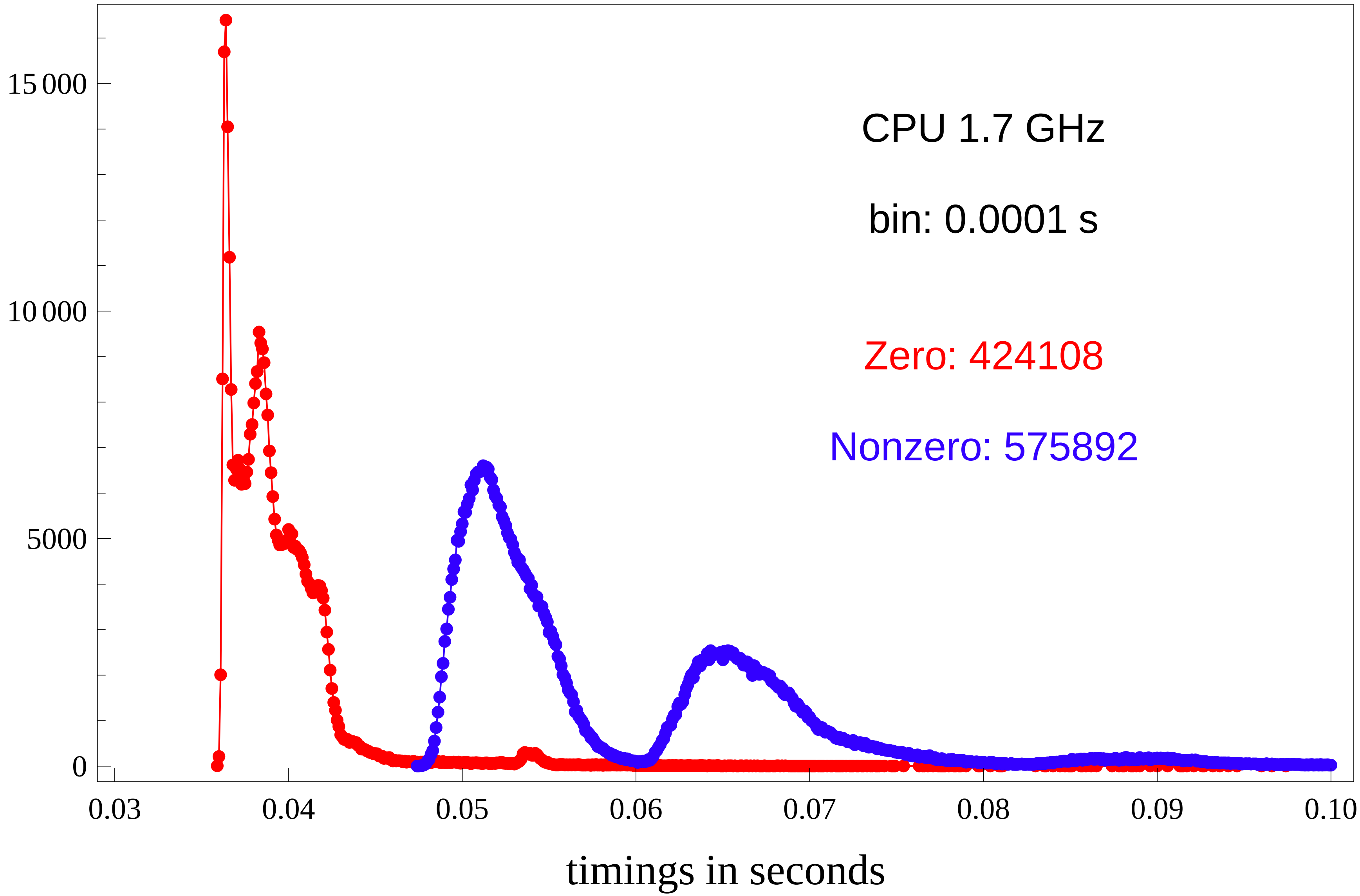
2e-4. Example 2

A million random monomial Riemann^7 scalars:



2e-5. Example 2

A million random monomial Riemann^{10} scalars:



3. The xAct project

3. The *xAct* project

xCore

Mathematica Tools

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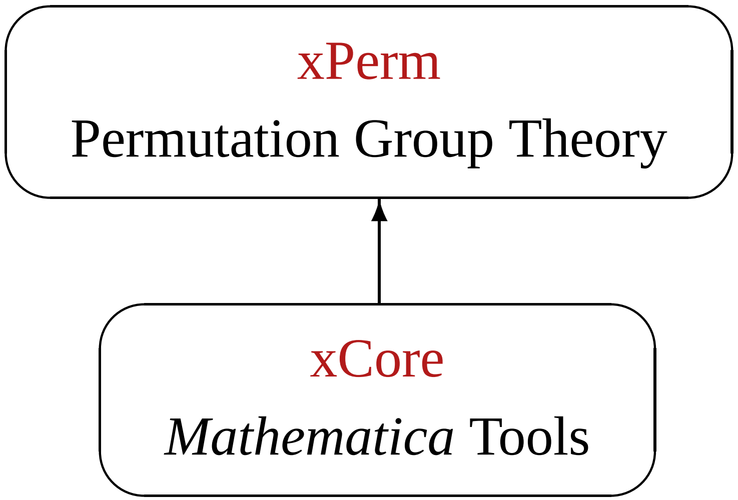

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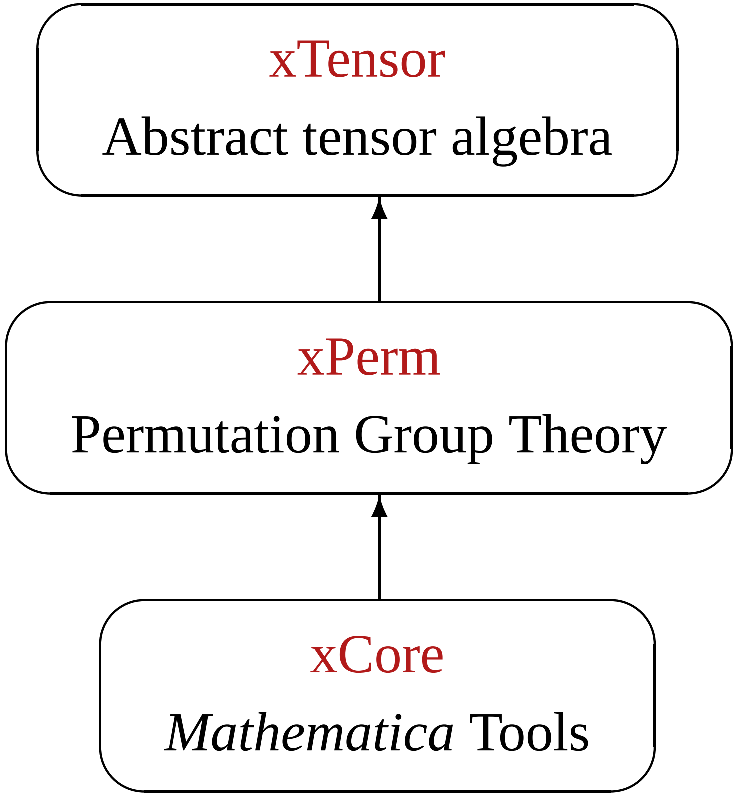

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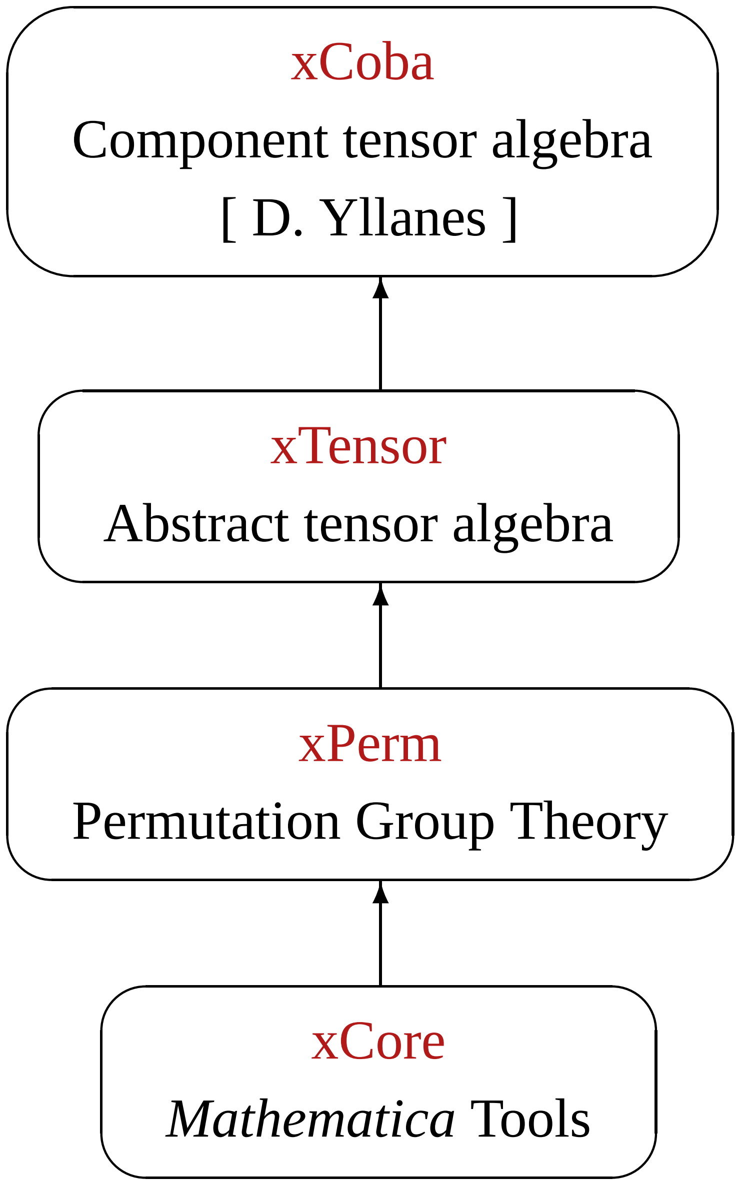

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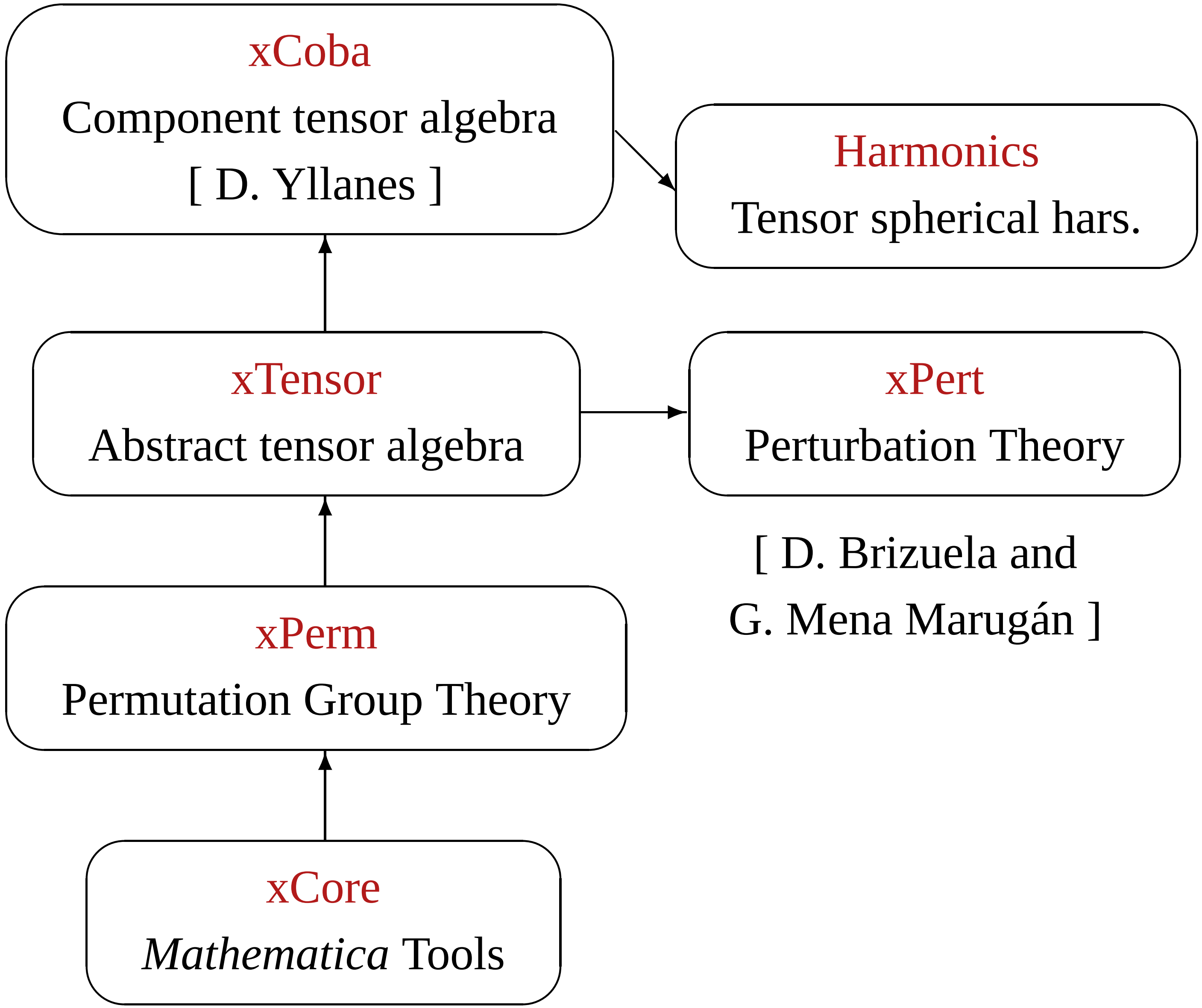

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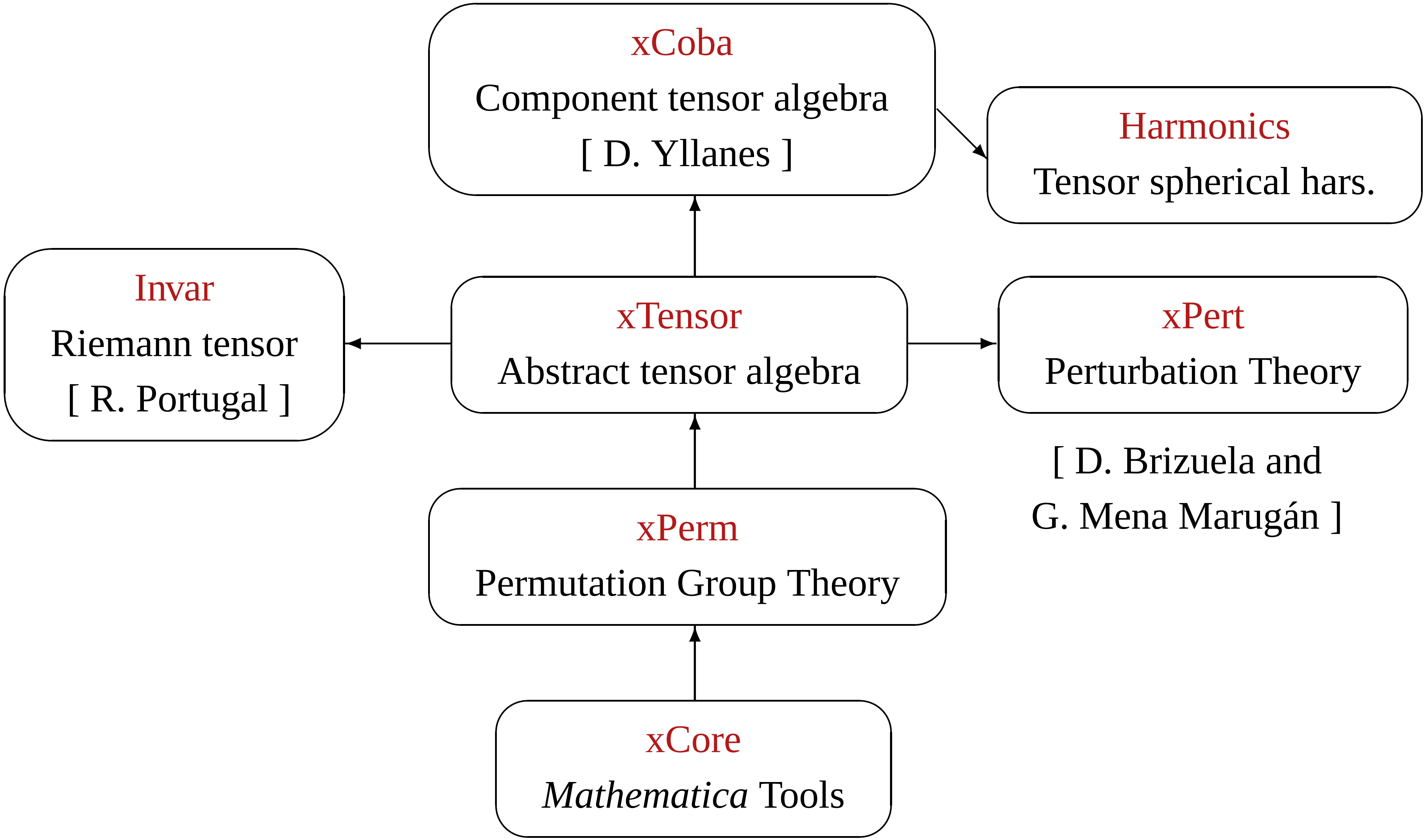

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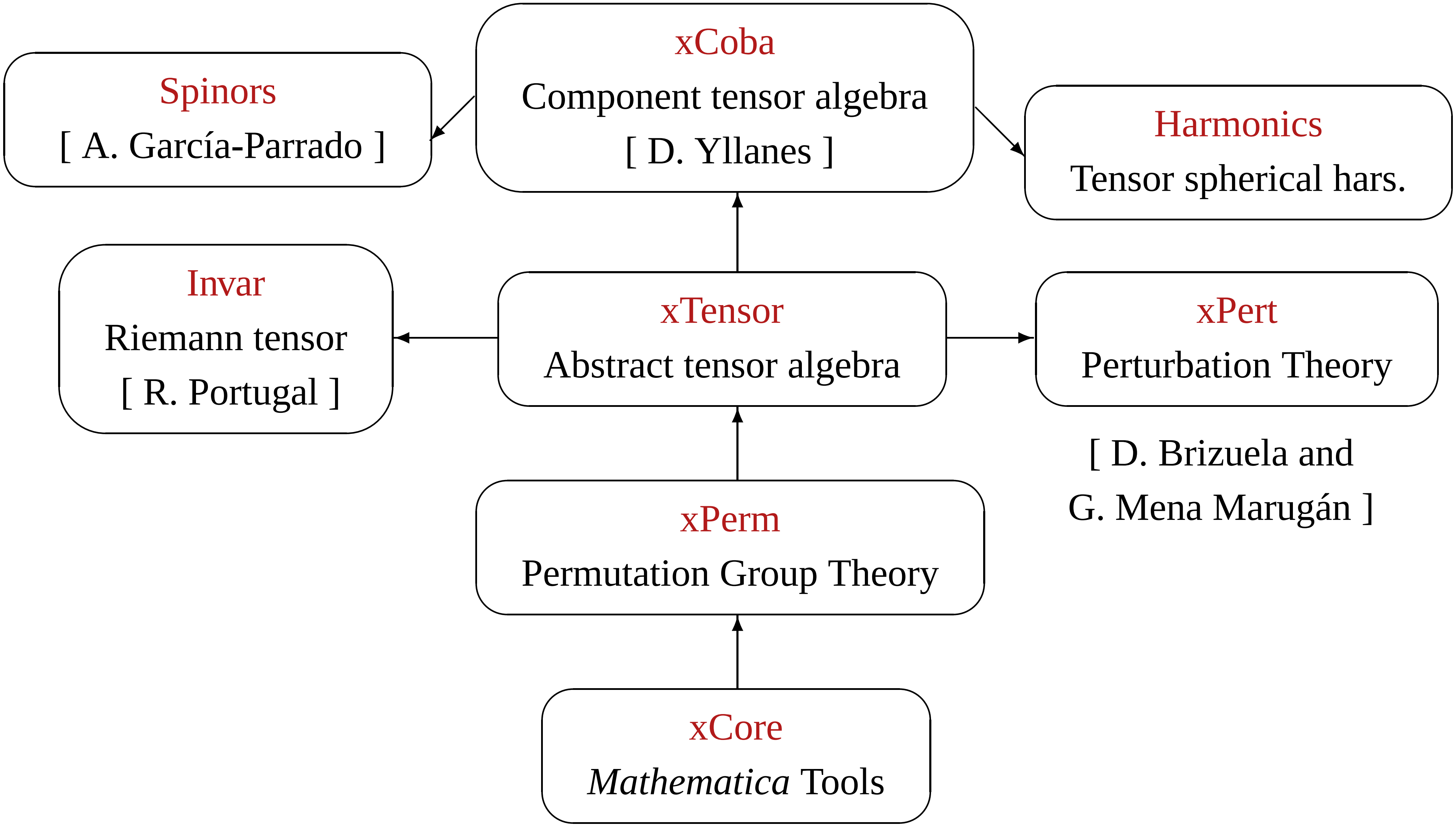

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- Mathematical structure, GR-oriented.
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Other data:

- ©2002–2009, GPL. Version 0.7 in March 2004; currently in 0.9.8
- 17000 lines of Mathematica code + 2500 lines of C code.
- 31 articles have used it:

3c. Results

- Hyperbolicity analysis of the Einstein equations (Gundlach & JMM)
- High order perturbation theory in GR (Brizuela & JMM)
- Invariants of the Riemann tensor (JMM & Portugal)
- The light-cone theorem (Choquet-Bruhat, Chruściel & JMM)
- Superfield integrals in string theory (Green et al.)
- Dynamical laws of superenergy (García-Parrado)
- Initial data sets for the Schwarzschild spacetime (GP & Valiente)
- Cosmological perturbation theory (Pitrou et al.)
- Post-Newtonian computations (Blanchet et al.)
- Quantum Field Theory (Álvarez et al.)
- “Galileon” (Deffayet et al.)

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- Define one or several manifolds, and products of them.
- Define (complex) vector bundles on them.
- Define tensors with arbitrary monoterm symmetries.
- Define connections of any type. Automatic Christoffel, Riemann, ... generation.
- Define one or several metrics on each manifold. Warped metrics. Induced metrics.

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  - Choice: ∇_a and ∂_a are operators: `Cd[-a][expr]`
- Lie derivatives: `LieD[ v[i] ][ expr ]`

4. The Riemann invariants

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5838550350981924166243854816386358011913196352080821446913112544207777945582431400973483457511551249
4271650405780114863779964319571108875740447236120954785394441817343113274871351581501474081446209153
5133993980627211654318697002059693685910102607365896788999230680327719504392651078493689021476459822
1917466623055176060658271638645490139036389024466375930303586688550738508615214422459534528028026604
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- We get the 27 invs of Sneddon's basis up to degree 7 (6 dual), plus all polynomial expression of any other invariant. Invar package: CPC 2007.
- Database of 645 625 relations up to 12 metric derivatives. CPC 2008.

4c. Riemann invariants

Degree	A	A*	B	B*	C	C*	D	D*
1	1	1	1	0	1	0	1	0
2	3	4	2	1	2	1	2	1
3	9	27	5	6	3	2	3	2
4	38	232	15	40	4	1	3	1
5	204	2582	54	330	5	2	3	2
6	1613	35090	270	3159	8	2	4	2
7	16532	558323	1639	–	7	(1)	3	(1)
8	217395	–	13140	–	(9)	(1)	(2)	(1)
9	3406747	–	–	–	(11)	(1)	(3)	(1)
10	–	–	–	–	(9)	(1)	(1)	(1)
11	–	–	–	–	(9)	(0)	(1)	(0)
12	–	–	–	–	(9)	(0)	(0)	(0)

A: Permutation symmetries, B: Cyclic symmetry,
C: Dim-dep identities, D: Products of duals (*).

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- There are multiterm algorithms, but we need something more efficient. Idea: databases of solutions (example: Invar package).
- xAct implements the fastest algorithms, in a GR-oriented structure based on Penrose abstract indices. Well tested and documented.

`http://metric.iem.csic.es/Martin-Garcia/xAct/`

`http://luth.obspm.fr/~luthier/Martin-Garcia/xAct/`